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BOOK 5

"MATHEMATICS, INFORMATICS AND PHYSICS"

VOLUME 9

CONTENTS

GRASSMANN ALGEBRA'S PI-PROPERTIES IN MATRIX ALGEBRAS WITH GRASSMANN ENTRIES

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***Abstract**: In the paper we consider some matrix algebras with Grassmann entries and investigate the "inheritance" by these algebras of some PI-properties of the Grassmann algebra. These properties include the Grassmann identity $[x, y, z] = 0$ and the nilpotency of the double commutator. The zero traces of some matrix expressions lead to new polynomial identities for special classes of matrix algebras with Grassmann entries.*

***Keywords:** Grassmann algebra, Grassmann identity, nilpotent commutator*

PRELIMINARIES

In the paper we work with the infinite dimensional Grassmann algebra E defined as

$$E = E(V) = K\langle e_1, e_2, \ldots \mid e_i e_j + e_j e_i = 0 i, j = 1,2,\ldots\rangle.$$

The field K has characteristic zero.

The algebra E is in the mainstream of recent research in PI-theory. Its importance is connected with the structure theory for the T-ideals of identities of associative algebras developed by Kemer [5]. Other examples concerning important applications of E could be found in [7].

The importance of considering matrix algebras $M_n(E)$ is confirmed by the following statement as the trivial isomorphism $E \otimes M_n(K) \cong M_n(E)$ holds:

Proposition 1 [4, Corollary 8.2.4, p. 111] *For every PI-algebra R there exists a positive n such that $T(R) \supseteq T(M_n(E))$, i.e. R satisfies all polynomial identities of the $n \times n$ matrix algebra $M_n(E)$ with entries from the Grassmann algebra.*

We list some well known facts concerning both the algebras E and $M_n(E)$ using [6,2,1].

Proposition 2 [6, Corollary, p. 437] *The T-ideal $T(E)$ is generated by the identity $[x_1, x_2, x_3] = 0$.*

Proposition 3 [2, Lemma 5.1] *The T-ideal $T(E)$ contains the identities*

$[x_1, x_2][x_3, x_4] + [x_1, x_3][x_2, x_4] = 0$ and $[x_1, x_2][x_2, x_3] = 0$.

Proposition 4 [1, Lemma 6.1] *The algebra E satisfies $S_n(x_1, \ldots, x_n)^k = 0$ for all $n, k \geq 2$ and*

$$S_n(x_1, \ldots, x_n) = \sum_{\sigma \in Sym(n)} (-1)^{\sigma} x_{\sigma(1)} \ldots x_{\sigma(n)}$$

being the standard identity.

Proposition 5 [1, Corollary 6.6] *The algebra $M_n(E)$ does not satisfy the identity $S_m(x_1, ..., x_m)^n = 0$ for any m.*

More facts concerning the PI-structure of E and $M_n(E)$ could be found in [4].

IDENTITIES FOR SOME SUBALGEBRAS OF $M_n(E)$

As Proposition 5 states the algebra $M_2(E)$ does not satisfy S_n^2 for any n i.e. $M_2(E)$ does not satisfy $[X,Y]^2=0$. Even there are such X and Y for which the commutator $[X,Y]$ is not nilpotent at all. Considering however some subalgebras of $M_2(E)$ (and of $M_n(E)$ for any n) we prove that $[X,Y]^k=0$ for some k, where X and Y are elements of these subalgebras.

Proposition 6 *There are elements X and Y from $M_2(E)$ for which $[X,Y]$ is not nilpotent.*

Proof: As an example we could take X and Y even from $M_2(K)$, namely $X=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$ and $Y=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ for which $[X,Y]=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ and $[X,Y]^2=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$.

Thus $[X,Y]^{2n-1}=(-1)^n\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ and $[X,Y]^{2n}=(-1)^n\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$.

The tensor product of E with commutative subalgebras of $M_2(K)$ inherits the identities of E. Here we give some concrete examples:

Proposition 7 *The algebra $SA1(E)=\{\begin{pmatrix}\alpha & \beta\\ \beta & \alpha\end{pmatrix}\}$ satisfies the Grassmann identity $[X,Y,Z]=0$ and the identity $[X,Y]^2=0$.*

Proof: Let $X,Y,Z\in SA1(E)$ be with entries $\alpha_i,\beta_i,i=1,2,3$, respectively. Then

$$[X,Y]=\begin{pmatrix}[\alpha_1,\alpha_2]+[\beta_1,\beta_2] & [\alpha_1,\beta_2]-[\alpha_2,\beta_1]\\ [\alpha_1,\beta_2]-[\alpha_2,\beta_1] & [\alpha_1,\alpha_2]+[\beta_1,\beta_2]\end{pmatrix}.$$

Forming $[X,Y]^2$ and applying Proposition 3 (namely that $[x_1,x_2]^2=0$) we get that

$$[X,Y]^2=2\begin{pmatrix}[\alpha_1,\alpha_2][\beta_1,\beta_2]-[\alpha_1,\beta_2][\alpha_2,\beta_1] & 0\\ 0 & [\alpha_1,\alpha_2][\beta_1,\beta_2]-[\alpha_1,\beta_2][\alpha_2,\beta_1]\end{pmatrix}.$$

The identity $[x_1,x_2][x_3,x_4]+[x_1,x_3][x_2,x_4]=0$ from Proposition 3 gives that $[\alpha_1,\beta_2][\alpha_2,\beta_1]=[\alpha_1,\alpha_2][\beta_1,\beta_2]$ and thus $[X,Y]^2=0$.

For the triple commutator $[X,Y,Z]=[a_{ij}]$ we get

$$a_{11}=a_{22}=[\alpha_1,\alpha_2,\alpha_3]+[\beta_1,\beta_2,\alpha_3]+[\alpha_1,\beta_2,\beta_3]-[\alpha_2,\beta_1,\beta_3],$$
$$a_{12}=a_{21}=[\alpha_1,\alpha_2,\beta_3]+[\beta_1,\beta_2,\beta_3]-[\alpha_3,[\alpha_1,\beta_2]]+[\alpha_3,[\alpha_2,\beta_1]].$$

Thus Proposition 2 gives that $[X,Y,Z]=0$.

Proposition 8 *The algebra* $SA2(E) = \{\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}\}$ *satisfies the Grassmann identity* $[X,Y,Z]=0$ *and the identity* $[X,Y]^2 = 0$.

Proof: In this case $[X,Y] = \begin{pmatrix} [\alpha_1,\alpha_2]-[\beta_1,\beta_2] & [\alpha_1,\beta_2]-[\alpha_2,\beta_1] \\ -[\alpha_1,\beta_2]+[\alpha_2,\beta_1] & [\alpha_1,\alpha_2]-[\beta_1,\beta_2] \end{pmatrix}$ and $[X,Y]^2 = -2\begin{pmatrix} [\alpha_1,\alpha_2][\beta_1,\beta_2]-[\alpha_1,\beta_2][\alpha_2,\beta_1] & 0 \\ 0 & [\alpha_1,\alpha_2][\beta_1,\beta_2]-[\alpha_1,\beta_2][\alpha_2,\beta_1] \end{pmatrix}$.

We could generalize the above two propositions for $n\times n$ matrices, namely

Proposition 9 *Any of the* $n\times n$ *matrix algebras* $\{\begin{pmatrix} \alpha_i & 0 & ... & 0 & \beta_i \\ 0 & ... & ... & ... & 0 \\ 0 & ... & ... & ... & 0 \\ ... & ... & ... & ... & ... \\ \beta_i & 0 & ... & 0 & \alpha_i \end{pmatrix}\}$,

$\{\begin{pmatrix} \alpha & \beta & ... & ... & ... & \beta \\ \beta & \alpha & \beta & ... & ... & \beta \\ \beta & \beta & \alpha & \beta & ... & \beta \\ ... & ... & ... & ... & ... & ... \\ \beta & ... & ... & ... & \beta & \alpha \end{pmatrix}\}$, $\{\begin{pmatrix} \alpha_i & 0 & ... & 0 & \beta_i \\ 0 & ... & ... & ... & 0 \\ 0 & ... & ... & ... & 0 \\ ... & ... & ... & ... & ... \\ -\beta_i & 0 & ... & 0 & \alpha_i \end{pmatrix}\}$ *and*

$\{\begin{pmatrix} \alpha & \beta & ... & ... & ... & \beta \\ -\beta & \alpha & \beta & ... & ... & \beta \\ -\beta & -\beta & \alpha & \beta & ... & \beta \\ ... & ... & ... & ... & ... & ... \\ -\beta & ... & ... & ... & -\beta & \alpha \end{pmatrix}\}$ satisfies the Grassmann identity $[X,Y,Z]=0$ and the identity $[X,Y]^2 = 0$.

Proof: We sketch the proof only for the second listed algebra. In this case for $[X,Y]=[a_{ij}]$ we have

$$a_{ii} = [\alpha_1,\alpha_2]+(n-1)[\beta_1,\beta_2],$$
$$a_{ij} = [\alpha_1,\beta_2]+[\beta_1,\alpha_2]+(n-2)[\beta_1,\beta_2].$$

For $[X,Y]^2 = [b_{ij}]$ we get

$$b_{ii} = 2(n-1)([\alpha_1,\alpha_2][\beta_1,\beta_2]+[\alpha_1,\beta_2][\beta_1,\alpha_2]),$$
$$b_{ij} = 0.$$

Proposition 10 *The algebra* $SA3(E) = \{\begin{pmatrix} 0 & 0 \\ \alpha & \beta \end{pmatrix}\}$ *satisfies the identities*

$[X,Y]^2 Z = 0$ and $[X,Y,Z]U = 0$ for any elements X,Y,Z,U of $SA3(E)$.

Proof: In this case $[X,Y] = \begin{pmatrix} 0 & 0 \\ \beta\alpha_1 - \beta_1\alpha & [\beta,\beta_1] \end{pmatrix}$ and applying Proposition 3 we get that $[X,Y]^2 = \begin{pmatrix} 0 & 0 \\ (\beta\alpha_1 - \beta_1\alpha)[\beta,\beta_1] & 0 \end{pmatrix}$ and $[X,Y,Z] = \begin{pmatrix} 0 & 0 \\ \alpha_2 - \beta_2(\beta\alpha_1 - \beta_1\alpha) & 0 \end{pmatrix}$.

This proposition is valid as well for $n \times n$ matrices of the considered type.

Some tensor products of E with noncommutative subalgebras of $M_2(K)$ satisfy identities "similar" to some of the identities of E i.e these are degrees of the identities of E.

We give an example here.

Proposition 11 *The algebra* $SA4(E) = \{\begin{pmatrix} \alpha & \alpha \\ \beta & \beta \end{pmatrix}\}$ *has nilpotent double commutators with index of nilpotency* ≤ 4.

Proof: For $A = \begin{pmatrix} \alpha & \alpha \\ \beta & \beta \end{pmatrix}$ and $B = \begin{pmatrix} \alpha_1 & \alpha_1 \\ \beta_1 & \beta_1 \end{pmatrix}$ we form

$$[A,B] = \begin{pmatrix} [\alpha,\alpha_1] + \alpha\beta_1 - \alpha_1\beta & [\alpha,\alpha_1] + \alpha\beta_1 - \alpha_1\beta \\ [\beta,\beta_1] + \beta\alpha_1 - \beta_1\alpha & [\beta,\beta_1] + \beta\alpha_1 - \beta_1\alpha \end{pmatrix}.$$

Then we evaluate the entries of the matrix $[A,B]^2 = [a_{ij}]$:

$$a_{11} = 2[\alpha,\alpha_1](\alpha\beta_1 - \alpha_1\beta) + (\alpha\beta_1 - \alpha_1\beta)^2 + [\alpha,\alpha_1][\beta,\beta_1]$$
$$+ [\alpha,\alpha_1](\beta\alpha_1 - \beta_1\alpha) + (\alpha\beta_1 - \alpha_1\beta)[\beta,\beta_1] + (\alpha\beta_1 - \alpha_1\beta)(\beta\alpha_1 - \beta_1\alpha).$$

Grouping the first and the forth summand, the second and the sixth one, we get consequently, using the properties of E:

$$a_{11} = [\alpha,\alpha_1](2\alpha\beta_1 - 2\alpha_1\beta + \beta\alpha_1 - \beta_1\alpha)$$
$$+ (\alpha\beta_1 - \alpha_1\beta)(\alpha\beta_1 - \alpha_1\beta + \beta\alpha_1 - \beta_1\alpha)$$
$$+ [\beta,\beta_1]([\alpha,\alpha_1] + \alpha\beta_1 - \alpha_1\beta)$$
$$= [\alpha,\alpha_1](\alpha\beta_1 + [\alpha,\beta_1] + [\beta,\alpha_1] - \alpha_1\beta)$$
$$+ (\alpha\beta_1 - \alpha_1\beta)([\alpha,\beta_1] + [\beta,\alpha_1])$$
$$+ [\beta,\beta_1]([\alpha,\alpha_1] + \alpha\beta_1 - \alpha_1\beta)$$
$$= [\alpha,\alpha_1](\alpha\beta_1 - \alpha_1\beta)$$
$$+ (\alpha\beta_1 - \alpha_1\beta)([\alpha,\beta_1] + [\beta,\alpha_1])$$
$$+ [\beta,\beta_1]([\alpha,\alpha_1] + \alpha\beta_1 - \alpha_1\beta)$$
$$= [\beta,\beta_1][\alpha,\alpha_1] + (\alpha\beta_1 - \alpha_1\beta)([\beta,\beta_1] + [\alpha,\alpha_1] + [\alpha,\beta_1] + [\beta,\alpha_1]).$$

With similar transformations we come to

$$a_{21} = [\beta,\beta_1][\alpha,\alpha_1] + [\beta,\beta_1](\alpha\beta_1 - \alpha_1\beta) + (\beta\alpha_1 - \beta_1\alpha)[\alpha,\alpha_1]$$
$$+ (\beta\alpha_1 - \beta_1\alpha)(\alpha\beta_1 - \alpha_1\beta) + [\beta,\beta_1](\beta\alpha_1 - \beta_1\alpha)$$
$$+ (\beta\alpha_1 - \beta_1\alpha)[\beta,\beta_1] + (\beta\alpha_1 - \beta_1\alpha)^2$$

$$= [\beta, \beta_1][\alpha, \alpha_1] + [\beta, \beta_1]([\alpha, \beta_1] + [\beta, \alpha_1])$$
$$+ (\beta\alpha_1 - \beta_1\alpha)([\alpha, \alpha_1] + [\beta, \beta_1])$$
$$+ (\beta\alpha_1 - \beta_1\alpha)(\alpha\beta_1 - \alpha_1\beta + \beta\alpha_1 - \beta_1\alpha)$$
$$= [\beta, \beta_1][\alpha, \alpha_1] + (\beta\alpha_1 - \beta_1\alpha)([\beta, \beta_1] + [\alpha, \alpha_1] + [\alpha, \beta_1] + [\beta, \alpha_1]).$$

For $[A, B]^3 = [b_{ij}]$ we get that

$$b_{11} = b_{12} = 2[\alpha, \alpha_1][\beta, \beta_1](\alpha\beta_1 - \alpha_1\beta)$$
$$+ (\alpha\beta_1 - \alpha_1\beta)^2([\beta, \beta_1] + [\alpha, \alpha_1] + [\alpha, \beta_1] + [\beta, \alpha_1])$$
$$+ 2[\alpha, \alpha_1][\beta, \beta_1](\beta\alpha_1 - \beta_1\alpha)$$
$$+ (\alpha\beta_1 - \alpha_1\beta)(\beta\alpha_1 - \beta_1\alpha)([\beta, \beta_1] + [\alpha, \alpha_1] + [\alpha, \beta_1] + [\beta, \alpha_1])$$
$$= 2[\alpha, \alpha_1][\beta, \beta_1]([\alpha, \beta_1] + [\beta, \alpha_1])$$
$$+ (\alpha\beta_1 - \alpha_1\beta)([\alpha, \beta_1] + [\beta, \alpha_1])([\beta, \beta_1] + [\alpha, \alpha_1] + [\alpha, \beta_1] + [\beta, \alpha_1])$$
$$= 2(\alpha\beta_1 - \alpha_1\beta)[\alpha, \beta_1][\beta, \alpha_1].$$

Analogously $b_{21} = b_{22} = 2(\beta\alpha_1 - \beta_1\alpha)[\alpha, \beta_1][\beta, \alpha_1]$.

Now it is easy to see, applying Proposition 3, that $[A, B]^4 = [c_{ij}] = 0$ as

$$c_{11} = c_{12} = 2(\alpha\beta_1 - \alpha_1\beta)([\alpha, \beta_1] + [\beta, \alpha_1])[\alpha, \beta_1][\beta, \alpha_1];$$
$$c_{21} = c_{22} = 2(\beta\alpha_1 - \beta_1\alpha)([\alpha, \beta_1] + [\beta, \alpha_1])[\alpha, \beta_1][\beta, \alpha_1].$$

It is interesting to define the traces of the commutator and of its nonzero degrees, namely

$$Tr[X, Y] = [\alpha, \alpha_1] + [\beta, \beta_1] + [\alpha, \beta_1] + [\beta, \alpha_1] = [\alpha + \beta, \alpha_1 + \beta_1];$$
$$Tr[X, Y]^2 = 2[\alpha, \alpha_1][\beta, \beta_1] + ([\alpha, \beta_1] + [\beta, \alpha_1])[\alpha + \beta, \alpha_1 + \beta_1]$$
$$= 2[\alpha, \alpha_1][\beta, \beta_1] + [\alpha, \beta_1][\beta, \alpha_1] + [\beta, \alpha_1][\alpha, \beta_1]$$
$$= 2([\alpha, \alpha_1][\beta, \beta_1] + [\alpha, \beta_1][\beta, \alpha_1]) = 0;$$
$$Tr[X, Y]^3 = 2([\alpha, \beta_1] + [\beta, \alpha_1])[\alpha, \beta_1][\beta, \alpha_1] = 0.$$

Proposition 12 *The algebra* $SA4(E) = \{\begin{pmatrix} \alpha & \alpha \\ \beta & \beta \end{pmatrix}\}$ *satisfies the identity* $[X, Y, Z]^2 = 0$.

Proof: Considering the matrix Z with entries α_2, β_2 accordingly and the notations from Proposition 11, we form the elements of the matrix $[X, Y, Z] = [a_{ij}]$, namely

$$a_{11} = a_{12} = [\alpha, \alpha_1, \alpha_2] + [\alpha\beta_1 - \alpha_1\beta, \alpha_2]$$
$$+ \left([\alpha, \alpha_1] + \alpha\beta_1 - \alpha_1\beta\right)\beta_2 - \alpha_2\left([\beta, \beta_1] + \beta\alpha_1 - \beta_1\alpha\right)$$
$$= \alpha_2[\alpha_1, \beta] - \alpha_2[\alpha, \beta_1] + (\alpha\beta_1 - \alpha_1\beta)(\alpha_2 + \beta_2) + [\alpha, \alpha_1]\beta_2 - \alpha_2[\beta, \beta_1]$$
$$= \alpha_2[\alpha_1, \beta] + [\alpha, \alpha_1]\beta_2 - \alpha_2[\alpha + \beta, \beta_1] + (\alpha\beta_1 - \alpha_1\beta)(\alpha_2 + \beta_2);$$

$$a_{21}=a_{22}=[[\beta,\beta_1]+\beta\alpha_1-\beta_1\alpha,\beta_2]$$
$$+([\beta,\beta_1]+\beta\alpha_1-\beta_1\alpha)\alpha_2-\beta_2([\alpha,\alpha_1]+\alpha\beta_1-\alpha_1\beta)$$
$$=(\beta\alpha_1-\beta_1\alpha)(\alpha_2+\beta_2)+\beta_2[\beta_1,\alpha]-\beta_2[\beta,\alpha_1]+[\beta,\beta_1]\alpha_2-\beta_2[\alpha,\alpha_1]$$
$$=(\beta\alpha_1-\beta_1\alpha)(\alpha_2+\beta_2)+[\beta,\beta_1]\alpha_2+\beta_2[\beta_1,\alpha]-\beta_2[\alpha+\beta,\alpha_1].$$

We show that $a_{11}+a_{22}=0$, i.e. $Tr[X,Y,Z]=0$. Really

$$a_{11}+a_{22}=(\alpha_2+\beta_2)(\beta\alpha_1-\beta_1\alpha+\alpha\beta_1-\alpha_1\beta)$$
$$+[\beta,\beta_1]\alpha_2+\beta_2[\beta_1,\alpha]-\beta_2[\alpha+\beta,\alpha_1]+\alpha_2[\alpha_1,\beta]+[\alpha,\alpha_1]\beta_2-\alpha_2[\alpha+\beta,\beta_1]$$
$$=(\alpha_2+\beta_2)([\beta,\alpha_1]+[\alpha,\beta_1])+[\beta,\beta_1]\alpha_2+\beta_2[\beta_1,\alpha]-\beta_2[\alpha+\beta,\alpha_1]$$
$$+\alpha_2[\alpha_1,\beta]+[\alpha,\alpha_1]\beta_2-\alpha_2[\alpha+\beta,\beta_1]$$
$$=\alpha_2[\alpha,\beta_1]+\beta_2[\beta,\alpha_1]+[\beta,\beta_1]\alpha_2-\beta_2[\alpha,\alpha_1]$$
$$-\beta_2[\beta,\alpha_1]+[\alpha,\alpha_1]\beta_2-\alpha_2[\alpha,\beta_1]-\alpha_2[\beta,\beta_1]=[\alpha,\alpha_1,\beta_2]+[\beta,\beta_1,\alpha_2]=0.$$

The elements of $[X,Y,Z]^2=[b_{ij}]$ are $b_{11}=b_{12}=a_{11}(a_{11}+a_{22})$ and $b_{21}=b_{22}=a_{22}(a_{11}+a_{22})$. Thus we get that $[X,Y,Z]^2=0$.

We come to the following

Corollary 1 The algebra $SA4(E)$ satisfies the identities $X[Y,Z,U]=0$ and $X[Y,Z]^2=0$.

Proof: For $A,B\in SA4(E)$ and $TrA=0$ the identity $BA=0$ holds.

It is easy to see that the last two propositions and Corollary 1 could be naturally generalized for $n\times n$ matrices.

Corollary 2 Let consider the algebra $SA5(E)$ of the $n\times n$ matrices having k rows with entries all equal to α and $n-k$ rows with entries all equal to β. Then $Tr[X,Y]^2=0$ and $Tr[X,Y,Z]=0$ for $X,Y,Z\in SA5(E)$.

Proof: We consider the first k rows with entries all equal to α. Using the notations of Proposition 11 we get that the two different entries $a_{11}=....=a_{kn}$ and $a_{k+1,1}=...=a_{nn}$ of the matrix $[A,B]$ are:

$$a_{11}=k[\alpha,\alpha_1]+(n-k)(\alpha\beta_1-\alpha_1\beta)$$
$$a_{nn}=(n-k)[\beta,\beta_1]+k(\beta\alpha_1-\beta_1\alpha).$$

For the two different entries $b_{11}=....=b_{kn}$ and $b_{k+1,1}=...=b_{nn}$ of the matrix $[A,B]^2$ we get

$$b_{11}=a_{11}(ka_{11}+(n-k)a_{nn})=a_{11}a$$
$$b_{nn}=a_{nn}(ka_{11}+(n-k)a_{nn})=a_{nn}a.$$

For the expression a we have

$$a=k^2[\alpha,\alpha_1]+k(n-k)(\alpha\beta_1-\alpha_1\beta)$$
$$+(n-k)^2[\beta,\beta_1]+(n-k)k(\beta\alpha_1-\beta_1\alpha)$$
$$=k^2[\alpha,\alpha_1]+(n-k)^2[\beta,\beta_1]+k(n-k)([\alpha,\beta_1]+[\beta,\alpha_1]).$$

Using the properties of the algebra E for the trace of the matrix $[A,B]^2$ we get that

$$Tr[A,B]^2 = kb_{11} + (n-k)b_{nn} = k(k(n-k)^2[\alpha,\alpha_1][\beta,\beta_1]) + (n-k)k(\alpha\beta_1 - \alpha_1\beta)a$$
$$+ (n-k)((n-k)k^2[\alpha,\alpha_1][\beta,\beta_1]) + (n-k)k(\beta\alpha_1 - \beta_1\alpha)a$$
$$= 2k^2(n-k)^2[\alpha,\alpha_1][\beta,\beta_1] + k(n-k)(\alpha\beta_1 - \alpha_1\beta)a + (n-k)k(\beta\alpha_1 - \beta_1\alpha)a$$
$$= 2k^2(n-k)^2[\alpha,\alpha_1][\beta,\beta_1] + k(n-k)([\alpha,\beta_1] + [\beta,\alpha_1])a$$
$$= 2k^2(n-k)^2[\alpha,\alpha_1][\beta,\beta_1] + k^2(n-k)^2[\alpha,\beta_1][\beta,\alpha_1] + k^2(n-k)^2[\alpha,\beta_1][\beta,\alpha_1]$$
$$= 2k^2(n-k)^2([\alpha,\alpha_1][\beta,\beta_1] + [\alpha,\beta_1][\beta,\alpha_1]) = 0.$$

Now we consider the matrix $[A,B,C] = [c_{ij}]$ having only two different entries $c_{11} = \dots = c_{kn}$ and $c_{k+1,1} = \dots = c_{nn}$. We do the following transformations:

$$c_{11} = k(k[\alpha,\alpha_1,\alpha_2] + (n-k)[\alpha\beta_1 - \alpha_1\beta,\alpha_2])$$
$$+ (n-k)\{(k[\alpha,\alpha_1] + (n-k)(\alpha\beta_1 - \alpha_1\beta)\beta_2 - \alpha_2((n-k)[\beta,\beta_1] + k(\beta\alpha_1 - \beta_1\alpha)\}$$
$$= k(n-k)(\alpha\beta_1 - \alpha_1\beta)\alpha_2 - k(n-k)\alpha_2(\alpha\beta_1 - \alpha_1\beta) + k(n-k)[\alpha,\alpha_1]\beta_2$$
$$+ (n-k)^2(\alpha\beta_1 - \alpha_1\beta)\beta_2 - (n-k)^2\alpha_2[\beta,\beta_1] - k(n-k)\alpha_2(\beta\alpha_1 - \beta_1\alpha).$$

$$c_{11} = k(n-k)\alpha_2([\alpha_1,\beta] - [\alpha,\beta_1]) + (n-k)(\alpha\beta_1 - \alpha_1\beta)(k\alpha_2 + (n-k)\beta_2)$$
$$+ k(n-k)[\alpha,\alpha_1]\beta_2 - (n-k)^2\alpha_2[\beta,\beta_1].$$

Analogously for c_{nn} we get that

$$c_{nn} = (n-k)\{(n-k)[\beta,\beta_1,\beta_2] + k[\beta\alpha_1 - \beta_1\alpha,\beta_2]\}$$
$$+ u\{((n-k)[\beta,\beta_1] + k(\beta\alpha_1 - \beta_1\alpha))\alpha_2 - \beta_2(k[\alpha,\alpha_1] + (n-k)(\alpha\beta_1 - \alpha_1\beta))\}$$
$$= k(n-k)\beta_2([\alpha_1,\beta] - [\alpha,\beta_1]) + k(\beta\alpha_1 - \beta_1\alpha)(k\alpha_2 + (n-k)\beta_2)$$
$$+ k(n-k)[\beta,\beta_1]\alpha_2 - k^2\beta_2[\alpha,\alpha_1].$$

Then for $Tr[A,B,C] = kc_{11} + (n-k)c_{nn}$ we get

$$Tr[A,B,C] = k(n-k)([\alpha_1,\beta] + [\beta_1,\alpha])(k\alpha_2 + (n-k)\beta_2)$$
$$+ k(n-k)([\alpha,\beta_1] + [\beta,\alpha_1])(k\alpha_2 + (n-k)\beta_2)$$
$$+ k(n-k)(k[\alpha,\alpha_1]\beta_2 - (n-k)\alpha_2[\beta,\beta_1] + (n-k)[\beta,\beta_1]\alpha_2 - k\beta_2[\alpha,\alpha_1])$$
$$= k(n-k)(k[\alpha,\alpha_1,\beta_2] + (n-k)[\beta,\beta_1,\alpha_2]) = 0.$$

Corollary 3 Let us consider the algebra $SA6(E)$ of $2n \times 2n$ matrices having n rows with entries all equal to α and n rows with entries all equal to β. The following identities hold: $X[Y,Z,U] = 0$ and $X[Y,Z]^2 = 0$.

Proof: In this case the matrices $[X,Y,Z]$ and $[Y,Z]^2$ have columns whose entries could be paired in a way $(1,-1)$ and multiplying on the left with any matrix from $SA6(E)$ we get the zero matrix.

REMARK

The paper marks a special stage of author's life and scientific activity.

REFERENCES

[1] Berele A. and A. Regev, Exponential growth for codimensions of some P.I. algebras, J. Algebra, Volume 241, (2001), 118-145.
[2] Drensky V., Free Algebras and PI-Algebras, Springer-Verlag, 1999.
[3] Drensky V. and E. Formanek, Polynomial Identity Rings, Birkhauser Verlag, 2004.
[4] Giambruno A. and M. Zaicev, Polynomial Identities and Assymptotic Methods, Math. Surveys and Monographs, Volume 122, American Mathematical Society, 2005.
[5] Kemer A.R., Ideals of Identities of Associative Algebras, Trans. Math. Monogr., Volume 87, American Mathematical Society, 1991.
[6] Krakowski D. and A. Regev, The polynomial identities of the Grassmann algebra, Trans. Amer. Math. Soc., Volume 181, (1973), 429-438.
[7] Rashkova Ts., On the nilpotency in matrix algebras with Grassmann entries, Serdica Math. J., Volume 38, (2012), in print.

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PI – СВОЙСТВА НА ГРАСМАНОВАТА АЛГЕБРА В МАТРИЧНИ АЛГЕБРИ С ГРАСМАНОВИ ЕЛЕМЕНТИ

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Резюме: *В статията се разглеждат матрични алгебри с грасманови елементи и се изучава "наследяването" от тези алгебри на някои PI-свойства на Грасмановата алгебра. Тези свойства включват Грасмановото тъждество* $[x,y,z]=0$ *и нилпотентността на двойния комутатор. Нулевата среда на някои матрични изрази води до нови полиномни тъждества за специални класове матрични алгебри с Грасманови елементи.*

Ключови думи: *Грасманова алгебра, Грасманово тъждество, нилпотентен комутатор.*

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