PROCEEDINGS

of the Union of Scientists - Ruse

Book 5

Mathematics, Informatics and Physics

Volume 10, 2013

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BOOK 5

"MATHEMATICS, INFORMATICS AND PHYSICS"

VOLUME 10

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BOOK 5

"MATHEMATICS, INFORMATICS AND PHYSICS"

VOLUME 10

This is the jubilee 10-th volume of book 5 Mathematics, Informatics and Physics. The beginning was in Spring, 2001, when the colleagues of the former section Mathematics and Physics decided to start publishing our own book of the Proceedings of the Union of Scientists – Ruse. The first volume included 24 papers. Through the years there have been authors not only from the Angel Kanchev University of Ruse but as well as from universities of Gabrovo, Varna, Veliko Tarnovo and abroad – Russia, Greece and USA.

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IDENTITIES OF $M_2(E)$ ARE IDENTITIES FOR CLASSES OF SUBALGEBRAS OF $M_n(E)$ AS WELL

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Abstract: *Vishne gave in [13] the explicit form of two polynomial identities of degree 8 for the matrix algebra $M_2(E)$, where E is the Grassmann algebra. In the paper we show that some subalgebras of $M_n(E)$ for arbitrary $n > 2$ satisfy these two identities as well. By a programme written in the system Mathematica a computer testing the value of the corresponding polynomials is done for one of the considered subalgebras in the case $n = 3$ over the finite dimensional Grassmann algebra E_4.*

Keywords: *Grassmann algebra, T-ideal, standard polynomial, multilinear identities, pattern.*

PRELIMINARIES

Let E denote the infinite dimensional Grassmann algebra on a countable dimension vector space over a field K of characteristic zero. A common presentation of E is the following one:

$$E = \langle e_1, e_2, \ldots \mid e_i e_j + e_j e_i = 0,\, i, j = 1,2,\ldots \rangle.$$

We list some well known facts concerning the algebra E:

Proposition 1. [8, Corollary, p. 437] *The T-ideal $T(E)$ is generated by the identity $[x_1, x_2, x_3] = [[x_1, x_2], x_3] = 0$, where $[x_1, x_2] = x_1 x_2 - x_2 x_1$.*

Proposition 2. [2, Lemma 6.1] *The algebra E satisfies $S_n(x_1, \ldots, x_n)^k = 0$ for all $n, k \geq 2$, where $S_n(x_1, \ldots, x_n)$ is the standard polynomial of degree n.*

The algebra E is in the mainstream of resent research in PI theory. Its importance is connected with the structure theory for the T-ideals of identities of associative algebras developed by Kemer. In [7, Theorem 1.2] he proved that any T-prime T-ideal can be obtained as the T-ideal of identities of one of three algebras, among which is the algebra $M_n(E)$.

Basic results, concerning both the algebras E and $M_n(E)$ could be found in [3,4,6].

In [1, p.356] some open questions in PI theory were stated. We mention two of them:

- Describe the identities of minimal degree of $M_n(E)$.
- Find a set of identities that generate the T-ideal of the identities of $M_n(E)$.

A. Popov and U. Vishne put the beginning of the investigations on the topic. In [10] Popov proved that the algebra $M_n(E)$ has no identities of degree $4n-2$. In [13] Vishne described an efficient way to use the $Sym(n)$-module structure of the ideal of multilinear identities in the computation of polynomial dentities of degree n of a given algebra. The method was used to show that $M_2(E)$ has identities of degree 8, but of no smaller degree. Two explicit identities of degree 8 were shown in [13].

In [9] using the system *Mathematica* a computer proof of the result of Vishne was given by Rashkova and Mihova.

Here we consider classes of subalgebras of $M_n(E)$, introduced in [11, 5]. They all satisfy the identity $X_4[X_1, X_2, X_3] = 0$. It appeared that if the identity $X_4[X_1, X_2, X_3] = 0$ holds in an algebra, the algebra satisfies the identities of degree 8, described by Vishne in [13], too. The proof of this statement is given below.

At the beginning we need some details from [13] for describing the considered multilinear polynomials of degree 8.

A *pattern* is a finite sequence of the letters A, B. If π is a pattern with a appearances of A and b of B, we denote by $\pi(x_1,\dots,x_a;y_1,\dots,y_b)$ the product of variables where the $x's$ and $y's$ are combined according to π. For example $ABBA(x_1,x_2;y_1,y_2) = x_1y_1y_2x_2$. A coefficient in front of a pattern π means that the monomial should be multiplied by that coefficient.

Let

$$P_\pi^+ = \sum_{\sigma\in Sym(a),\tau\in Sym(b)} sign(\sigma)\pi(x_{\sigma(1)},\dots,x_{\sigma(a)};y_{\tau(1)},\dots,y_{\tau(b)}),$$

$$P_\pi^- = \sum_{\sigma\in Sym(a),\tau\in Sym(b)} sign(\sigma)sign(\tau)\pi(x_{\sigma(1)},\dots,x_{\sigma(a)};y_{\tau(1)},\dots,y_{\tau(b)})$$

and

$$\mathrm{P} = \begin{pmatrix} +AAAABAAB, & +AABBAAAA, & -AABAAAAB, \\ -AAAABBAA, & -BAABAAAA, & +BAAAABAA \end{pmatrix}.$$

We construct the polynomial

$$T_1(x_1,\dots,x_6;y_1,y_2) = \sum_{\pi\in\mathrm{P}} (P_\pi^- + P_\pi^+). \tag{1}$$

In it only the monomials with y_1 preceeding y_2 appear.

For

$$\mathrm{PP} = \begin{pmatrix} -AAABAABB, & -AABBAABA, & +ABBAABAA, \\ +AAABBAAB, & +AABAABBA, & -ABAABBAA \\ -ABBAAAAB, & +BAABBAAA, & -BAAAABBA \\ +ABAAAABB, & -BBAABAAA, & +BBAAAABA \end{pmatrix}$$

we consruct analogously

$$T_2(x_1,\dots,x_5;y_1,y_2,y_3) = \sum_{\pi\in\mathrm{PP}} (P_\pi^- + P_\pi^+). \tag{2}$$

The polynomial T_2 has only the monomials in which the order of y_1, y_2, y_3 is even.

Proposition 3. [13, Corollary 4.2] *T_1 and T_2 are multilinear identities of degree 8 of $M_2(E)$.*

Now we give some examples of subalgebras of $M_n(E)$, considered in [11, 5], satisfying an identity of degree 4.

Proposition 4. [11, Corollary 3] The subalgebra of $M_{2n}(E)$ of $2n\times 2n$ matrices having n rows with entries all equal to α and n rows with entries all equal to β satisfies the identity $X_4[X_1, X_2, X_3] = 0$.

Proposition 5. [5, Theorems 4, 5, 7] The following algebras

A1=($\begin{pmatrix} x_1 & \dots & x_1 & kx_1 & \dots & kx_1 \\ x_2 & \dots & x_2 & kx_2 & \dots & kx_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ x_{2n} & \dots & x_{2n} & kx_{2n} & \dots & kx_{2n} \end{pmatrix}, x_i \in E, k \in K$),

A2=($\begin{pmatrix} y_1 & y_2 & \dots & y_n \\ \alpha_2 y_1 & \alpha_2 y_2 & \dots & \alpha_2 y_n \\ \dots & \dots & \dots & \dots \\ \alpha_n y_1 & \alpha_n y_2 & \dots & \alpha_n y_n \end{pmatrix}, y_j \in E, \alpha_k \in K$) and

A3=($\begin{pmatrix} z_1 & 0 & \dots & \dots & \dots & 0 & z_1 \\ 0 & z_2 & 0 & \dots & 0 & z_2 & 0 \\ \dots. & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & z_{n+1} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & z_{2n} & 0 & \dots & 0 & z_{2n} & 0 \\ z_{2n+1} & 0 & \dots & \dots & \dots & 0 & z_{2n+1} \end{pmatrix}, z_s \in E$)

satisfy the identity $X_4[X_1, X_2, X_3] = 0$.

THE THEOREM AND ITS PROOF

Here we prove the following

Theorem. Any algebra $\mathbb{A}$, satisfying the identity $x_4[x_1, x_2, x_3] = 0$, satisfies as well the identities $T_1(x_1, \dots, x_6; y_1, y_2) = 0$ and $T_2(x_1, \dots, x_5; y_1, y_2, y_3) = 0$ from (1) and (2), respectively.

Proof of the Theorem:

At the beginning we give a way of presenting the identity of degree 4 by means of the pattern notation. Really the identity $0 = y_1[x_1, x_2, y_2]$ could be written as

$$\begin{aligned} 0 &= y_1[x_1, x_2]y_2 - y_1 y_2[x_1, x_2] \\ &= \sum_{\sigma \in Sym(2), \pi = BAAB} sign(\sigma)\pi(x_{\sigma(1)} x_{\sigma(2)}; y_1, y_2) \\ &- \sum_{\sigma \in Sym(2), \pi = BBAA} sign(\sigma)\pi(x_{\sigma(1)} x_{\sigma(2)}; y_1, y_2). \end{aligned}$$

For short we use the notation

$$BAAB - BBAA = 0. \quad (3)$$

I. We write P as a 2×3 matrix $P = (a_{ij})$.

Thus we get $a_{11} + a_{21} = AAAA(BAAB - BBAA) = 0$. This really means that the polynomial, corresponding to the pattern $BAAB-BBAA$, is multiplied on the left by x_3, after two consequitive substituions (123) of the indices of x's we multiply the corresponding sum on the left by x_4, again substitutions of the indices (1234) take place, multiplication on the left by x_5 follows, circular substitution of the indices of x's takes place, multiplication on the left by x_6 follows and a circular substitution of the the indices of x's finishes the operations.

The result is that $\sum_{\pi=AAAABAAB-AAAABBAA}(P_\pi^- + P_\pi^+)=0$.

The next analogous consequences of the identity $0=y_1[x_1,x_2,y_2]$ we'll be skeched in the same formal way.

If in (3) we substitute the first B by $AABAA$, we get $AABAAAAB-AABAABAA=AABAAAAB-AABBAAAA$, as (3) holds, i.e. $a_{12}+a_{13}=0$. For $a_{22}+a_{23}=0$ we substitute in (3) the first B by BAA and multiply on the right the new identity by AA.

As a final result we get that $T_1(x_1,\ldots,x_6;y_1,y_2)=0$ is an identity for $\mathbf{A}$.

II. In order to prove the second identity we use again the formal notation $PP=(b_{ij})$ as a 4×3 matrix and the above shorter way of presenting the corresponding consequences of the identity $0=y_1[x_1,x_2,y_2]$.

Obviously

$$\begin{aligned} b_{12}+b_{22} &= AA(BAAB-BBAA)BA=0 \\ b_{32}+b_{42} &= (BAAB-BBAA)BAAA=0 \\ b_{11}+b_{21} &= -AAA(BAAB-BBAA)B=0 \\ b_{13}+b_{23} &= -A(BAAB-BBAA)BAA=0 \end{aligned}.$$

In (3) we substitute the first B by BAA and then multiply on the right by BA. At the end we have to change the indices of y's in order to cover the even substitutions of y_1,y_2,y_3. Thus we come to

$$b_{33}+b_{43}=(BAAAAB-BAABAA)BA=(BAAAAB-BBAAAA)BA=0.$$

At last we multiply (3) on the right by B, then we substitute the first B by $ABAA$. We get

$$b_{31}+b_{41}=(ABAAAAB-ABAABAA)B=(ABAAAAB-ABBAAAA)B=0.$$

The result is that $T_2(x_1,\ldots,x_5;y_1,y_2,y_3)=0$.

This ends the proof of the Theorem.

COMPUTER TESTING THE VALIDITY OF THE THEOREM

Some results in investigating properties either of the Grassmann algebra E or $M_n(E)$ were suggested or tested using a programme in the system of computer algebra *Mathematica* for manipulating with finite dimensional subalgebras E_k and the matrix algebra $M_2(E_k)$ for small k [9].

For the purposes of the paper this programme was modified for working in the 3×3 matrix algebra $A3(E_4)$ (the algebra $A3$ is defined in Proposition 5). The entries of the matrices from $A3(E_4)$ are written as vectors, each with 16-th coordinates. The notation $(\alpha_1,\alpha_2,...,\alpha_{16})$ stands for the Grassmann element

$$\alpha_1+\alpha_2e_1+\alpha_3e_2+\alpha_4e_1e_2+\alpha_5e_3+\alpha_6e_1e_3+\alpha_7e_2e_3+\alpha_8e_1e_2e_3+\alpha_9e_4+\alpha_{10}e_1e_4$$
$$+\alpha_{11}e_2e_4+\alpha_{12}e_1e_2e_4+\alpha_{13}e_3e_4+\alpha_{14}e_1e_3e_4+\alpha_{15}e_2e_3e_4+\alpha_{16}e_1e_2e_3e_4.$$

The coefficients α_i for the nonzero entries are random integers of a preliminary given interval.

The programme calculates the value of the polynomial B_{12} (corresponding to the pattern $BAAB-BBAA$) and those of T_1 и T_2 from (1) and (2), respectively. The computer realization of the last two polynomials is done recurrently using 35 operators for T_1 and 60 for T_2. The details are given in [12]. Here we give the presentation of a matrix in $A3(E_4)$ and some final values only:

```
x1={{Array[a1,16],{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0},
Array[a1,16]},{{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0},Array[b1,16],
{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0}},{Array[c1,16],
{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0},Array[c1,16]}}
{{{20,-6,-12,17,-6,-4,4,11,-9,5,-14,-11,18,-18,-19,12},
{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0},
{20,-6,-12,17,-6,-4,4,11,-9,5,-14,-11,18,-18,-19,12}},
{{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0},
{1,-19,-14,-18,-2,15,12,19,7,0,19,4,-8,9,18,-6},
{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0}},{{4,-17,-10,13,19,0,20,4,-1,-
19,-17,15,7,4,3,-1},{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0},{4,-17,-
10,13,19,0,20,4,-1,-19,-17,15,7,4,3,-1}}}
```

```
B12[x1_,x2_,y1_,y2_]:=y2⊗(S2[x1,x2]⊗y1)-y2⊗(y1⊗S2[x1,x2])
B12[x1,x2,y1,y2]
{{{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0},{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0
,0},{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0}},{{0,0,0,0,0,0,0,0,0,0,0,0,0
,0,0,0},{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0},{0,0,0,0,0,0,0,0,0,0,0,0
,0,0,0,0}},{{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0},{0,0,0,0,0,0,0,0,0,0
,0,0,0,0,0,0},{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0}}}
```

```
T1[x1_,x2_,x3_,x4_,x5_,x6_,y1_,y2_]:=A[x1,x2,x3,x4,x5,x6,y1,y2]+
B[x1,x2,x3,x4,x5,x6,y1,y2]-CH[x1,x2,x3,x4,x5,x6,y1,y2]-
DH[x1,x2,x3,x4,x5,x6,y1,y2]-
EH[x1,x2,x3,x4,x5,x6,y1,y2]+FH[x1,x2,x3,x4,x5,x6,y1,y2]
```

The parts of the polynomial $T_1(x_1,\ldots,x_6;y_1,y_2)$ are defined in [12]. The correspondence with the steps in the proof of the theorem is the following: The polynomial $a_{11}+a_{21}$ is realized in *Mathematica* by the polynomial $A(x_1,\ldots,x_6;y_1,y_2)-DH(x_1,\ldots,x_6;y_1,y_2)$, the expression $a_{12}+a_{13}$ leads to $B(x_1,\ldots,x_6;y_1,y_2)-CH(x_1,\ldots,x_6;y_1,y_2)$ and the computer construction of $a_{22}+a_{23}$ is the polynomial $FH(x_1,\ldots,x_6;y_1,y_2)-EH(x_1,\ldots,x_6;y_1,y_2)$.

Here we give the result for $a_{12} + a_{13}$ only:

```
B[x1,x2,x3,x4,x5,x6,y1,y2]
{{{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,-
2438631710136000},{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0},{0,0,0,0,0,0,0
,0,0,0,0,0,0,0,0,-
2438631710136000}},{{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0},{0,0,0,0,0,0
,0,0,0,0,0,0,0,0,0,0},{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0}},{{0,0,0,0
,0,0,0,0,0,0,0,0,0,0,0,2438631710136000},{0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0},{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,2438631710136000}}}
```

```
CH[x1,x2,x3,x4,x5,x6,y1,y2]
{{{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,-
2438631710136000},{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0},{0,0,0,0,0,0,0
,0,0,0,0,0,0,0,0,-
2438631710136000}},{{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0},{0,0,0,0,0,0
,0,0,0,0,0,0,0,0,0,0},{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0}},{{0,0,0,0
,0,0,0,0,0,0,0,0,0,0,0,2438631710136000},{0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0},{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,2438631710136000}}}
```

Thus we get the final result for the first identity:

```
T1[x1,x2,x3,x4,x5,x6,y1,y2]
{{{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0},{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0
,0},{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0}},{{0,0,0,0,0,0,0,0,0,0,0,0,0
,0,0,0},{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0},{0,0,0,0,0,0,0,0,0,0,0,0
,0,0,0,0}},{{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0},{0,0,0,0,0,0,0,0,0,0
,0,0,0,0,0,0},{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0}}}
```

The second identity is tested analogously. The polynomials considered in the proof of the theorem correspond to respective parts of the polynomial $T_2(x_1,\dots,x_5;y_1,y_2,y_3)$.

The programme ran on a computer AMD Athlon(tm) II X4 630 with 3,25 GB of RAM.

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ТЪЖДЕСТВА ЗА $M_2(E)$ СА ТЪЖДЕСТВА ЗА ПОДАЛГЕБРИ НА $M_n(E)$

Цецка Рашкова

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Резюме: *В [13] Вишне дава явната форма на две полиномни тъждества от степен* 8 *за матричната алгебра* $M_2(E)$*, където* E *е Грасмановата алгебра. В статията се доказва, че те са тъждества и за някои подалгебри на* $M_n(E)$ *при произволно* $n>2$*. Чрез програма на системата Mathematica е направено компютърно пресмятане стойността на съответните полиноми за една от разглежданите матрични алгебри при* $n=3$ *над крайномерната Грасманова алгебра* E_4*.*

Ключови думи*: Грасманова алгебра,* T*-идеал, стандартен полином, полилинейни тъждества, мотив.*

ISSN 1314-3077