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BOOK 5

"MATHEMATICS, INFORMATICS AND PHYSICS"

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"MATHEMATICS, INFORMATICS AND PHYSICS"

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MEAN VALUE THEOREMS IN DISCRETE CALCULUS

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Abstract: *Discrete analogues of some basic theorems of differential calculus such as Rolle's theorem, Lagrange's and Cauchy's Theorems, L'Hospital's rule are presented and proved in the paper. The discrete Lagrange's theorem is applied to an eigenvalue problem for p - difference equations.*

Keywords*: Discrete functions, Rolle's theorem, Cauchy's theorem, Mean value theorem, finite differences, difference calculus.*

INTRODUCTION

In recent years there has been an increasing interest in calculus of finite differences and discrete boundary value problems. There are several reasons for this. The advent of high speed computers and the need of a technique for (approximate) differentiation of functions employing arithmetic operations only have led to a need for fundamental knowledge of difference calculus. The modelling of certain nonlinear problems from biological neural networks, economics, optimal control and other areas of study have led to the rapid development of the theory of difference equations (see the monographs [1], [2]).

There is a remarkable analogy of the theory of finite differences to that of differential and integral calculus and differential equations. Note, however, that quantum results hold in general for functions which are necessarily continuous unlike in discrete calculus. In view of the importance of the Mean Value Theorem (MVT) in numerical methods and theory of critical points we are going to discuss the discrete versions of the basic theorems of differential calculus, such as Rolle's theorem, Cauchy's theorem and L'Hospital's Rule. We present an application of the discrete version of the MVT to p- difference equations.

NOTATIONS AND DEFINITIONS.

We shall use the following notations: $N_0 = \{0,1,2,3...\}$ the set of natural numbers including zero, $[a, b] = \{a, a+1, a+2, ..., b-1, b\}$, where $a, b \in N_0$. Let $x(t)$ be a discrete real function of the variable $t \in N_0$. The forward difference operator Δ is defined by

$$\Delta x(t) = x(t+1) - x(t),$$

and the backward difference operator ∇ is defined by

$$\nabla x(t) = x(t) - x(t-1) = \Delta x(t-1).$$

We also define the operator $\Delta_p x(t) = \Delta \varphi_p x(t)$, where $\varphi_p(t) = t\,|t|^{p-2}$, $p > 1$. It is obvious that $\varphi_p(t)$ has the following properties:

1. $\varphi_p(t_1 t_2) = \varphi_p(t_1)\varphi_p(t_2)$.
2. $\varphi_p(\lambda\, t) = \varphi_p(\lambda)\varphi_p(t) = \begin{cases} \lambda^{p-1}\varphi_p(t), & \lambda > 0 \\ -\mu^{p-1}\varphi_p(t), & \lambda = -\mu < 0. \end{cases}$
3. $\Delta_p x(t) = \Delta\varphi_p x(t) = \varphi_p x(t+1) - \varphi_p x(t)$.

Local extrema. Discrete Analogues of the Mean Value Theorems.

Let $x(t)$ be a discrete real function defined on $[a, b] \subset N_0$, $a > b$ and $\{k-1, k, k+1\} \subset [a+1, b-1]$ be three consecutive natural numbers.

Definition 1. The number $k \in [a+1, b-1]$ is said to be a *critical point* of $x(t)$ if

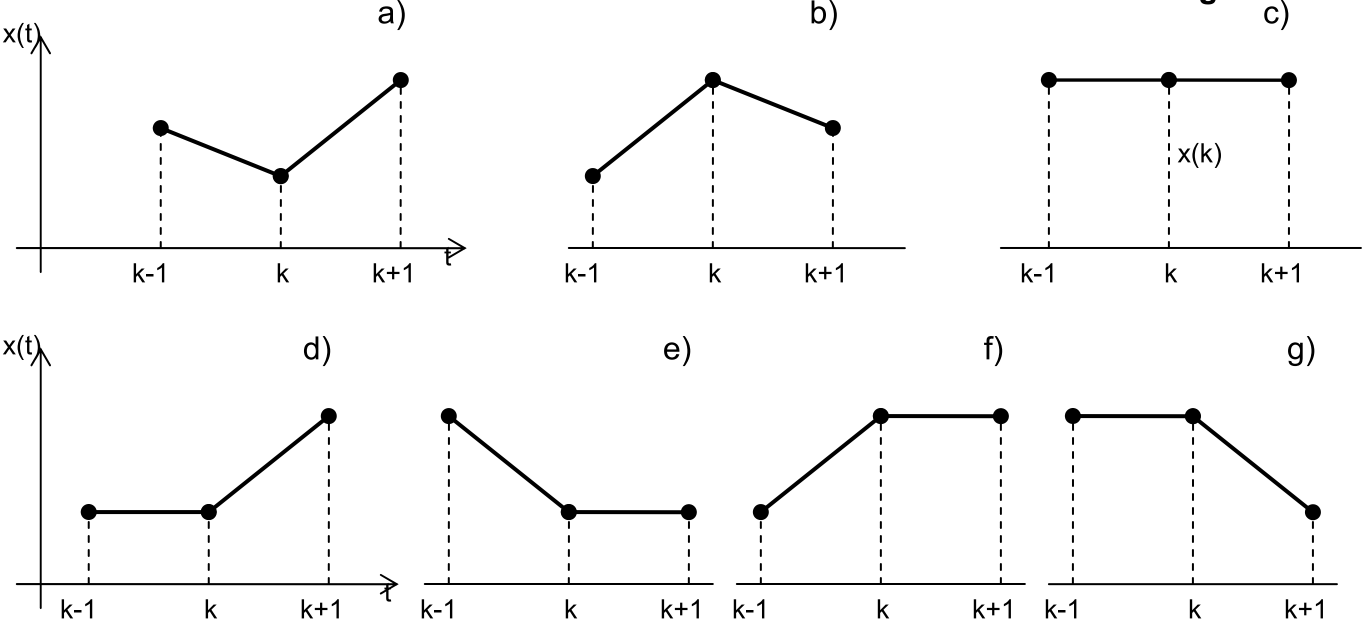
(1) $\Delta x(k)\,\Delta x(k-1) \le 0$, i.e. $\Delta x(k)\nabla x(k) \le 0$.

Definition 2. The number $k \in [k-1,\ k+1] \subset [a+1,\ b-1]$ is said to be a *local maximum (or local minimum)* of the function $x(t)$, defined on $[a,\ b] \subset N_0$ if

(2) $x(k) \ge \max\{x(k-1),\ x(k+1)\}$ (resp. $x(k) \le \min\{x(k-1),\ x(k+1)\}$).

The local maximum (local minimum) is said to be strict, if inequality (2) is strict. Hence, when k is a local extremum, which means local maximum or local minimum, inequality (1) holds, i.e. k is a critical point of $x(t)$.

Figure 1



In the case of:

- strict extremum (figure 1 - a, b), (1) holds with strict inequality;
- local minimum (figure 1 - d, e) and local maximum (figure 1 - f, g) one of the factors of (1) equals zero and in case 1-c both of them are equal to zero.

Theorem 1. Discrete analogue of Rolle's theorem. If $x(a) = x(b)$ and $x(t)$ is a scalar function defined on $[a,\ b] = \{a,\ a+1,\ a+2,...,\ b-1,\ b\} \subset N_0$, then there exists a local extremum point $k \in [a+1,\ b-1]$, where $b-a \ge 2$.

Proof: Let us assume the contrary statement is true, i.e. $x(a) = x(b)$ and there exists no local extremum point $k \in [a+1,\ b-1]$. Then $\Delta x(t)\,\Delta x(t-1) > 0$ for every $t = a+1,\ a+2,...,\ b-1$. Hence, any two consecutive differences have the same sign:

$$\big(x(a+2) - x(a+1)\big)\big(x(a+1) - x(a)\big) > 0,$$
$$\big(x(a+3) - x(a+2)\big)\big(x(a+2) - x(a+1)\big) > 0,$$
$$\dots\dots\dots\dots$$
$$\big(x(b) - x(b-1)\big)\big(x(b-1) - x(b-2)\big) > 0.$$

Let $x(a+1) - x(a) > 0$.

Then

$$x(a+2) - x(a+1) > 0,$$
$$x(a+3) - x(a+2) > 0,$$
$$\dots\dots\dots$$
$$x(b-1) - x(b-2) > 0,$$
$$x(b) - x(b-1) > 0.$$

Summing the above inequalities, we find $x(b)-x(a)>0$, i.e. $x(b)>x(a)$. In a similar way, in case $x(a+1)-x(a)<0$ we obtain $x(b)<x(a)$. We get a contradiction with the condition $x(a)=x(b)$ of the assumption. Thus there exists a local extremum point $k\in[a+1,\ b-1]$.□

Definition 3. The point $k\in[a+1,\ b-1]$ is said to be a *local* p*-extremum point* of the function $x(t),\ t\in[a,\ b]\subset N_0$ if

$$\Delta_p x(k)\,\Delta_p x(k-1)\le 0.$$

Lemma . Let $x(t)$ be a function defined on $[a,\ b]\subset N_0$. A point $k\in[a+1,\ b-1]$ is a local p-extremum point of the function if and only if k is a local extremum point of $x(t)$.

Proof. Let k be a local p-extremum point of the function $x(t)$ and the inequality $\Delta_p x(k)\,\Delta_p x(k-1)\le 0.$ holds, i.e.

$$(\varphi_p x(k+1)-\varphi_p x(k))\,(\varphi_p x(k)-\varphi_p x(k-1))\le 0.$$

<u>Case 1:</u> Let $\varphi_p x(k+1))-\varphi_p x(k)\ge 0$ and $\varphi_p x(k)-\varphi_p x(k-1)\le 0$. Since φ_p is a strictly increasing function it follows that $x(k+1)\ge x(k)$ and $x(k)\le x(k-1)$. Hence $\Delta x(k)\,\Delta x(k-1)\le 0$.

<u>Case 2:</u> $\varphi_p x(k+1))-\varphi_p x(k)\le 0$ and $\varphi_p x(k)-\varphi_p x(k-1)\ge 0$. Similarly to the first case we get $\Delta x(k)\ \Delta x(k-1)\le 0$.□

Likewise we prove the reversed statement.

Corollary 1. Let $x(t):[a,\ b]\to R$ be a discrete function and $x(a)=x(b)$. Then there exists a p-local extremum point $k\in[a+1,\ b-1]$.

Theorem 2. Mean Value Theorem (Discrete analogue of Lagrange's theorem)
Let $x(t)$ be a discrete valued function, defined on $[a,\ b]\subset N_0$. Then there exists a $k\in[a+1,\ b-1]$, such that

$$\left(\Delta x(k)-\frac{x(b)-x(a)}{b-a}\right)\left(\Delta x(k-1)-\frac{x(b)-x(a)}{b-a}\right)\le 0. \tag{3}$$

Note, that the latter inequality implies that either

$$\Delta x(k)\ge\frac{x(b)-x(a)}{b-a}\ge\Delta x(k-1)\ \text{ or }\ \Delta x(k)\le\frac{x(b)-x(a)}{b-a}\le\Delta x(k-1).$$

Proof. Let

$$v(k)=x(k)-\frac{x(b)-x(a)}{b-a}\left(k-a\right).$$

Then $v(a)=v(b)=x(a)$. By Rolle's theorem there exists $k\in[a+1,\ b-1]$, such that $\Delta v(k)\,\Delta v(k-1)\le 0$. Hence result (3) follows.□

Corollary 2. Let $x(t)$ be a discrete function, defined on $[a,\ b]\subset N_0$ and $M=\max\{|\,\Delta x(k)\,|\}$, where $k\in[a,\ b-1]$. Then,

$$\left|\frac{x(b)-x(a)}{b-a}\right|\le M.$$

Theorem 3. Discrete analogue of Cauchy's theorem. Let $x(t),\ y(t),\ t\in[a,\ b]$ be two discrete functions, and $y(t)$ be strictly monotonous for any $t\in[a,\ b-1]$, ($\Delta y(t)>0$ for any t or $\Delta y(t)<0$). Then there exists $k\in[a+1,\ b-1],$ such that

$$\left(\frac{\Delta x(k)}{\Delta y(k)}-\frac{x(b)-x(a)}{y(b)-y(a)}\right)\left(\frac{\Delta x(k-1)}{\Delta y(k-1)}-\frac{x(b)-x(a)}{y(b)-y(a)}\right)\le 0,\text{ i.e.}$$

$$\frac{\Delta x(k-1)}{\Delta y(k-1)}\le\frac{x(b)-x(a)}{y(b)-y(a)}\le\frac{\Delta x(k)}{\Delta y(k)},$$

or

$$\frac{\Delta x(k-1)}{\Delta y(k-1)}\ge\frac{x(b)-x(a)}{y(b)-y(a)}\ge\frac{\Delta x(k)}{\Delta y(k)}.$$

Proof. Let us denote $A=\frac{x(b)-x(a)}{y(b)-y(a)}$ and define an additional function

$$w(t)=x(t)-A(y(t)-y(a)).$$

Then $w(a)=w(b)=x(a)$. By Rolle's theorem there exists $k\in[a+1,\, b-1],$ such that

$$\big(\Delta x(k)-A\Delta y(k)\big)\big(\Delta x(k-1)-A\Delta y(k-1)\big)\le 0.$$

Dividing both sides by $\Delta y(k)\Delta y(k-1)>0$ we get

$$\left(\frac{\Delta x(k)}{\Delta y(k)}-A\right)\left(\frac{\Delta x(k-1)}{\Delta y(k-1)}-A\right)\le 0.$$

Thus,

$$\left(\frac{\Delta x(k)}{\Delta y(k)}-\frac{x(b)-x(a)}{y(b)-y(a)}\right)\left(\frac{\Delta x(k-1)}{\Delta y(k-1)}-\frac{x(b)-x(a)}{y(b)-y(a)}\right)\le 0.$$

Theorem 4. Discrete L'Hospital's Rule.

Let $x(k),\ \ y(t),\ \ t\in[1,\infty)$ be two discrete functions, such that

$$\lim_{t\to\infty}x(t)=\lim_{t\to\infty}y(t)=0$$

and assume there exists a number n_0, such that for any $n\ge n_0$

(4) $$\Delta y(n)<0,\quad y(n)>0.$$

Then, if the limit $\lim_{t\to\infty}\frac{\Delta x(t)}{\Delta y(t)}=c$ exists, it follows that $c=\lim_{t\to\infty}\frac{x(t)}{y(t)}$.

Proof. Let $\varepsilon>0$ and $n\in N$. Take $n_\varepsilon>n$ such that $|x(n_\varepsilon)|<\varepsilon$ and $|y(n_\varepsilon)|<\varepsilon$ $n\in N$. By Cauchy's theorem there exists $k_n\in[n+1,\, n_\varepsilon-1]$:

$$\left(\frac{\Delta x(k_n)}{\Delta y(k_n)}-\frac{x(n)-x(n_\varepsilon)}{y(n)-y(n_\varepsilon)}\right)\left(\frac{\Delta x(k_n-1)}{\Delta y(k_n-1)}-\frac{x(n)-x(n_\varepsilon)}{y(n)-y(n_\varepsilon)}\right)\le 0.$$

By $|x(n_\varepsilon)|<\varepsilon$, $|y(n_\varepsilon)|<\varepsilon$, inequalities (4) and because $k_n\to\infty$ as $n\to\infty$, we have that

$$\frac{x(n)-\varepsilon}{y(n)+\varepsilon}\le\frac{x(n)-x(n_\varepsilon)}{y(n)-y(n_\varepsilon)}\le\frac{\Delta x(k_n)}{\Delta y(k_n)}\to c,\ \text{as}\ n\to\infty.$$

Since $\varepsilon>0$ is arbitrary small, we get

(5) $$\overline{\lim_{n\to\infty}}\frac{x(n)}{y(n)}\le c.$$

Similarly,

$$\frac{x(n)+\varepsilon}{y(n)-\varepsilon}\ge\frac{x(n)-x(n_\varepsilon)}{y(n)+y(n_\varepsilon)}\ge\frac{\Delta x(k_n-1)}{\Delta y(k_n-1)}\to c,\ \text{as}\ n\to\infty.$$

Hence,

(6) $$\underline{\lim_{n\to\infty}}\frac{x(n)}{y(n)}\ge c.$$

Thus, using inequalities (5) and (6) we obtain

$$c \le \varliminf_{n\to\infty} \frac{x(n)}{y(n)} \le \varlimsup_{n\to\infty} \frac{x(n)}{y(n)} \le c \quad \Rightarrow \quad \lim_{n\to\infty} \frac{u(n)}{v(n)} = c.$$

AN APPLICATION OF MEAN VALUE THEOREM TO p- DIFFERENCE EQUATION.

We apply the discrete version of the Mean Value Theorem to an eigenvalue problem of a p - difference equation. The second order quasilinear difference equation for $p+1$ is studied in [3].

Consider the eigenvalue problem

(7) $$\Delta\left(\varphi_p\left(\Delta x(n-1)\right)\right) + \lambda\,\varphi_p\left(\Delta x(n)\right) = 0,$$

where $a, b \in N$, $n \in [a-1, b]$, $\varphi_p(t) = t\,|t|^{p-2}$, $p > 1$.

Our main result is

Theorem 5. Suppose that $\lambda > 0$ is an eigenvalue of the difference equation (7), with the corresponding nontrivial solution $\{x(n)\}$, $x(n) > 0$ for any $n \in [a, b-1]$, and either

(a) $\Delta x(a-1) \ge 0, \quad \Delta x(a) < 0$ and $x(b-1) > 0, \quad x(b) = 0,$ or

(b) $x(a-1) = 0, \quad x(a) > 0$ and $\Delta x(b-2) > 0, \quad \Delta x(b-1) \le 0.$

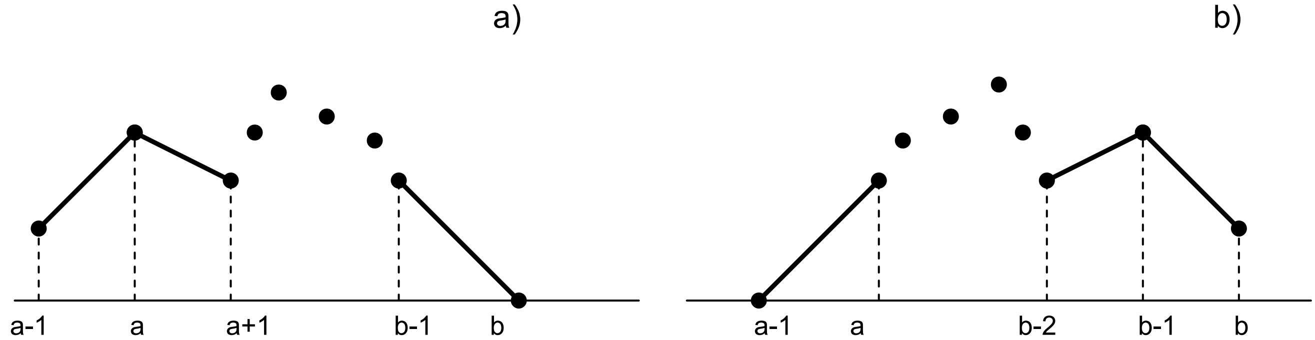
Then,

$$\lambda > \frac{1}{(b-a)^p}.$$

Proof. Suppose case (a) holds (Figure 2– a) and let

$$r = \max\{s:\ \Delta x(s-1) \ge 0,\ \ \Delta x(s) < 0,\ \ s \in [a, b-1]\}.$$

Figure 2



Hence, $\Delta x(n-1) < 0$ for $n \in [r+1, b]$, and $x(n) \ge 0$ for any $n \in [a, b]$. Then

(8) $$\Delta\left(\varphi_p\left(\Delta x(n-1)\right)\right) = -\lambda\,\varphi_p\left(\Delta x(n)\right) \le 0.$$

Since $\Delta\left(\varphi_p\left(\Delta x(n-1)\right)\right) = \varphi_p\left(\Delta x(n)\right) - \varphi_p\left(\Delta x(n-1)\right) \le 0$, it follows that $\Delta x(n) \le \Delta x(n-1)$ or $-\Delta x(n) \ge -\Delta x(n-1)$ and the sequence $\{\Delta x(n)\}$ is non-increasing. Since $\Delta x(n-1) < 0$, $x(n)$ is decreasing for $n \in [r, b]$.

By Lagrange's theorem there exists a $k \in [r+1, b]$,

$$\frac{-x(r)}{b-r} = \frac{x(b)-x(r)}{b-r} \ge \Delta x(k-1) \ge \Delta x(b-1),$$

which implies

$$\frac{x(r)}{b-r} \le -\Delta x(b-1).$$

Therefore

(9) $$\frac{(x(r))^{p-1}}{(b-r)^{p-1}} \le \left(-\Delta x(b-1)\right)^{p-1} = -\varphi_p\left(x(b-1)\right).$$

From (8) for any $n\in[r, b]$ and $x(n)\ge 0$, we have

$$\Delta\varphi_p\left(\Delta x(r-1)\right) = -\lambda\varphi_p\left(x(r)\right) = -\lambda\left(x(r)\right)^{p-1}$$

……………………………………………

$$\Delta\varphi_p\left(\Delta x(b-2)\right) = -\lambda\varphi_p\left(x(b-1)\right) = -\lambda\left(x(b-1)\right)^{p-1}$$

Summing the equalities above we obtain

$$\varphi_p\left(\Delta x(b-1)\right) - \varphi_p\left(\Delta x(r-1)\right) = -\lambda\sum_{k=r}^{b-1}\left(x(k)\right)^{p-1}, \text{ i.e.}$$

$$\varphi_p\left(\Delta x(r-1)\right) - \varphi_p\left(\Delta x(b-1)\right) = \lambda\sum_{k=r}^{b-1}\left(x(k)\right)^{p-1}.$$

Since $\Delta x(r-1)\ge 0$ and $\varphi_p(\Delta x(r-1)) = \Delta x(r-1)\left|\Delta x(r-1)\right|^{p-2}\ge 0$, we have

$$-\varphi_p\left(\Delta x(b-1)\right) < \lambda\sum_{k=r}^{b-1}\left(x(r)\right)^{p-1} = \lambda\left(x(r)\right)^{p-1}\sum_{k=r}^{b-1}1$$

$$= \lambda\left(x(r)\right)^{p-1}(b-1-r+1)$$

$$= \lambda\left(x(r)\right)^{p-1}(b-r),$$

(10) $$-\varphi_p\left(\Delta x(b-1)\right) < \lambda\left(x(r)\right)^{p-1}(b-r).$$

Now, by (9) and (10), we get

$$\frac{\left(x(r)\right)^{p-1}}{(b-r)^{p-1}} < -\varphi_p\left(\Delta x(b-1)\right) < \lambda\left(x(r)\right)^{p-1}(b-r).$$

Thus, we conclude that

$$\frac{1}{(b-a)^p} < \frac{1}{(b-r)^p} < \lambda. \square$$

REFERENCES

[1] Agarwal R.P., Difference Equations and Inequalities, Marcel Dekker, New York, 1992.

[2] Kelley W.G., A. C. Peterson, Difference equations. An introduction with applications, Academic Press, San Diego, 2001.

[3] Yong Zhou, Oscillation and nonoscillation criteria for second order quasilinear difference equations, J. Math Anal. Appl. 303 (2005), 365-375.

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ТЕОРЕМИ ЗА СРЕДНИТЕ СТОЙНОСТИ В ДИФЕРЕНЧНОТО СМЯТАНЕ

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Резюме: *В тази статия са представени и доказани дискретни аналози на някои основни теореми на диференциалното смятане - теоремите на Рол, Лагранж, Коши, Лопитал. Дискретният вариант на теоремата на Лагранж е приложен към задача за собствените стойности на p - диференчно уравнение.*

Ключови думи*: Дискретни функции, диференчно смятане, теорема на Рол, теорема на Коши, теореми за средните стойности.*

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