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CONTENTS

THREE APPROACHES TOWARD SUBLINEAR EMDEN – FOWLER EQUATION

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Abstract: *In this paper three different proofs for existence of positive solutions of the singular boundary value problem related to Emden—Fowler equation are presented. Variety of methods for investigation of differential equations are demonstrated such as the method of lower and upper solutions, variational methods, and a fixed point theorem for positive operators acting in a positive cone.*
Keywords: *singular boundary value problems,classical solutions, Theorem of Rothe, minimization of functionals*

INTRODUCTION

This paper is concerned with the existence of positive solutions of the generalized Emden – Fowler equation

$$u'' + p(t)u^{\lambda} = 0, \quad 0 < t < 1, \tag{1}$$

under the Dirichlet boundary conditions

$$u(0) = u(1) = 0 \tag{2}$$

in sublinear case $0 < \lambda < 1$. Here the coefficient $p \in C(0,1),\ p(t) \geq 0$ is singular since it might be unbounded at 0 and 1. Singular problems arise for example from processes in electrically conducting materials, chemical reactions in heterogeneous catalysis, and describe many other physical phenomena. In particular, the generalized Emden – Fowler equation appears in studies of fluid mechanics, nuclear physics, plasmas.

The aim of this paper is to present variety of techniques for investigation of differential equations. More precisely, we demonstrate three different proofs of the following existence theorem.

MAIN THEOREM *Suppose that*

$$0 < \int_0^1 t(1-t)p(t)dt < \infty. \tag{3}$$

Then problem (1)-(2) has at least one positive solution $u \in C[0,1] \cap C^2(0,1)$.

First proof given by Zhang [Z] uses the method of lower and upper solutions. Second proof obtained by Chaparova and Sanchez [CS] relies on variational approach. Third proof mentioned in Chaparova and Kutev [CK] is based on the theory of positive operators in Banach spaces acting in a positive cone.

FIRST PROOF BY LOWER AND UPPER SOLUTIONS

Following the paper of Zhang [Z], we recall the classical theorem concerning the method of lower and upper solutions for the nonsingular BVP

$$\begin{cases} u'' + f(t,u) = 0 \\ u(a) = r_1, \quad u(b) = r_2. \end{cases} \tag{4}$$

THEOREM ([BL]) *Suppose that* $\alpha, \beta \in C^2[a,b]$ *satisfy the following hypotheses:*

(i) $\alpha(t) \leq \beta(t)$ *for* $t \in [a,b]$,

(ii) $f(t,u)$ *is continuous on* $\Omega = \{(t,u) : t \in [a,b], \alpha(t) \le u \le \beta(t)\}$*;*

(iii) $\begin{cases} \alpha''(t) + f(t,\alpha(t)) \ge 0 \\ \beta''(t) + f(t,\beta(t)) \le 0 \end{cases} \quad t \in [a,b].$

Then for any real numbers r_1, r_2 *such that* $\alpha(a) \le r_1 \le \beta(a)$, $\alpha(b) \le r_2 \le \beta(b)$ *there exists at least one solution* $u \in C^2[a,b]$ *of (4) which satisfies* $\alpha(t) \le u(t) \le \beta(t)$ *for* $t \in [a,b]$.

The pair of functions $\alpha(t), \beta(t)$ in the previous theorem is referred to as the lower and upper solutions.

Now we present sketchily first proof of the Main Theorem.

FIRST PROOF Let us consider the functions $q_1(t), q_2(t)$ defined as follows.

$$q_1(t) = (1-t)\int_0^t s^{1+\lambda}(1-s)^{\lambda} p(s)ds + t\int_t^1 s^{\lambda}(1-s)^{1+\lambda} p(s)ds,$$

$$q_2(t) = (1-t)\int_0^t sp(s)ds + t\int_t^1 (1-s)p(s)ds.$$

Since $p \in C(0,1)$ satisfies (3), and

$$0 < t(1-t)\int_0^1 s^{1+\lambda}(1-s)^{1+\lambda} p(s)ds \le q_1(t) < q_2(t) \le \int_0^1 s(1-s)p(s)ds, \quad t \in (0,1) \qquad (5)$$

we have $q_1, q_2 \in C[0,1] \cap C^2(0,1)$, and

$$q_1'' = -t^{\lambda}(1-t)^{\lambda} p(t), \qquad q_2'' = -p(t),$$
$$q_1(0) = q_1(1) = 0, \qquad q_2(0) = q_2(1) = 0.$$

Indeed, in order to check that $\lim_{t \to 0^+} q_1(t)$ do exists and equals 0 we take into account the absolute convergence of the Lebesgue integral, i.e. (3) implies for every $\varepsilon > 0$ there is $\delta > 0$ such that $\int_0^{\delta} sp(s)ds < \varepsilon/2$. We denote $M = \int_{\delta}^1 (1-s)p(s)ds$. Then for $0 < t < \delta_1 = \min(\delta, \varepsilon/2M)$

$$t\int_t^1 s^{\lambda}(1-s)^{1+\lambda} p(s)ds = t\int_t^{\delta} s^{\lambda}(1-s)^{1+\lambda} p(s)ds + t\int_{\delta}^1 s^{\lambda}(1-s)^{1+\lambda} p(s)ds$$
$$< \int_t^{\delta} sp(s)ds + \delta_1\int_{\delta}^1 (1-s)p(s)ds < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$$

Let $\alpha(t) = L_1 q_1(t), \quad \beta(t) = L_2 q_2(t)$ where

$$L_1 = \left(\int_0^1 s^{1+\lambda} (1-s)^{1+\lambda} p(s) ds \right)^{\frac{\lambda}{1-\lambda}}, \qquad L_2 = \left(\int_0^1 s(1-s) p(s) ds \right)^{\frac{\lambda}{1-\lambda}}.$$

Thus $\alpha, \beta \in C[0,1] \cap C^2(0,1)$, $0 < \alpha(t) < \beta(t)$, $t \in (0,1)$, $\alpha(0) = \alpha(1) = \beta(0) = \beta(1) = 0$. By (5),

$$\begin{aligned} \alpha'' + p(t)\alpha^\lambda &= -L_1 t^\lambda (1-t)^\lambda p(t) + L_1^\lambda p(t) q_1^\lambda \\ &= L_1^\lambda p(t) \left[q_1^\lambda - L_1^{1-\lambda} t^\lambda (1-t)^\lambda \right] \geq 0, \quad t \in (0,1), \\ \beta'' + p(t)\beta^\lambda &= -L_1 p(t) + L_2^\lambda p(t) q_2^\lambda \\ &= L_2^\lambda p(t) \left[q_2^\lambda - L_2^{1-\lambda} \right] \leq 0, \quad t \in (0,1). \end{aligned}$$

Consider now the sequence of nonsingular BVPs over expanded subintervals of $(0,1)$,

$$\begin{cases} u'' + p(t)u^\lambda = 0, & t \in (a_n, b_n), \\ u(a_n) = x_n, \ u(b_n) = y_n \end{cases} \tag{6}$$

where $0 < \ldots < a_2 < a_1 < b_1 < b_2 < \ldots < 1$, $a_n \to 0$, $b_n \to 1$, and $\alpha(a_n) \leq x_n \leq \beta(a_n)$, $\alpha(b_n) \leq y_n \leq \beta(b_n)$. It is obvious that for each n $\alpha(t), \beta(t)$ is a pair of lower and upper solutions for problem (6). By the previous Theorem, we have a sequence of solutions $(u_n(t))$ such that $u_n(t) \in C^2[a_n, b_n]$.

By standard arguments, it can be shown that $(u_n(t))$ is uniformly bounded and equicontinuous on compact subintervals of $(0,1)$. Then Arzela–Ascoli theorem implies existence of uniformly convergent subsequence. The limit function $u(t)$ after continuation on $[0,1]$ is desired solution of the singular problem (1), (2). ■

SECOND PROOF BY VARIATIONAL TECHNIQUES

Consider first the nonsingular problem

$$\begin{cases} u'' + p(t)u^\lambda = 0, & t \in (0,1), \\ u(0) = 0, \ u(1) = 0 \end{cases} \tag{7}$$

where $p \in C[0,1]$, $p \geq 0$, $0 < \lambda < 1$. The problem has variational structure and its solutions can be obtained as critical points of the associated functional

$$I(u) = \int_0^1 \left(\frac{1}{2} u'^2 - \frac{1}{1+\lambda} p(t) |u|^\lambda u \right) dt$$

in Sobolev space $H_0^1(0,1)$ with the usual norm $\|u\| = \left(\int_0^1 u'^2(t) dt \right)^{1/2}$. Indeed, by definition $u \in H_0^1(0,1)$ is weak solution of (7) if

$$\int_0^1 \left(u'v' - p(t)|u|^{\lambda} v \right) dt = 0 \text{ for any } v \in H_0^1(0,1).$$

On the other hand, by standard argument $I \in C^1$, and

$$(I'(u), v) = \int_0^1 \left(u'v' - p(t)|u|^{\lambda} v \right) dt .$$

Thus $I'(u) = 0$ if and only if u is weak solution of (7). By the embedding theorem, weak solutions of (7) are also classical solutions.

In order to obtain critical points of I we apply the general minimization theorem.

MINIMIZATION THEOREM ([MW]) *Let the functional J be coercive bounded below and weak lower semicontinuous in the Banach space E. Then J has a minimum, i.e. there exists $u \in E$ such that $J(u) = \inf_{v \in E} J(v)$.*

LEMMA *The nonsingular problem (7) has at least one positive solution $u \in C^2[0,1]$ which is the minimizer of I.*

PROOF. Denote $|p|_{\infty} = \max_{t \in [0,1]} p(t)$, $|u|_{1+\lambda} = \left(\int_0^1 |u|^{1+\lambda} \right)^{1/(1+\lambda)}$ the norms in $C[0,1]$ and $L^{1+\lambda}(0,1)$ respectively. For any $u \in H_0^1(0,1)$ the inequality

$$|u(t)| \le \int_0^t |u'(s)| ds \le \sqrt{t} \|u\|$$

implies $|u|_{1+\lambda} \le \left(2/(3+\lambda)\right)^{1/(1+\lambda)} \|u\|$. Then

$$I(u) \ge \frac{1}{2}\|u\|^2 - \frac{1}{1+\lambda}|p|_{\infty} \left(|u|_{1+\lambda}\right)^{1+\lambda} \ge \frac{1}{2}\|u\|^2 - C\|u\|^{1+\lambda}, \quad u \in H_0^1(0,1).$$

Since $f(x) = (1/2)|x|^2 - C|x|^{1+\lambda}$ is bounded below and $f(x) \to +\infty$ as $x \to \infty$ the functional is bounded below and coercive. By standard arguments I is weakly lower semicontinuous. Thus it has a minimizer $u \in H_0^1(0,1)$ which is a nonnegative solution of (7).

In order to prove $u(t) > 0,\ t \in (0,1)$ let us take an arbitrary function $\bar{u} \in H_0^1(0,1)$ such that $\bar{u}(t) > 0,\ t \in (0,1)$. Then for $\xi > 0$ sufficiently small we have

$$I(\xi \bar{u}) = \frac{1}{2}\|\bar{u}\|^2 \xi^2 - \left(\frac{1}{1+\lambda} \int_0^1 p(t) \bar{u}^{1+\lambda}(t) dt \right) \xi^{1+\lambda} < 0$$

which means that $I(u) = \min_{v \in H_0^1(0,1)} I(v) \le I(\xi \bar{u}) < 0$, i.e. $u \ne 0$. ■

Now we turn to the singular problem (1), (2).

SECOND PROOF OF THE MAIN THEOREM The proof is rather technical that is why it will be discussed briefly. For details the reader is referred to [CS].

Consider the sequence of the approximated problems

$$(P_n):\begin{cases} u'' + p_n(t)u^{\lambda} = 0, & t \in (0,1), \\ u(0) = 0,\ u(1) = 0 \end{cases}$$

where $p_n \in C[0,1]$ are such that

$$p_n(t) = p(t), \quad t \in [a_n, b_n] \subset (0,1),$$
$$0 \le p_n(t) \le p_{n+1}(t) \le p(t), \quad t \in [0,1],$$

$0 < \ldots < a_2 < a_1 < b_1 < b_2 < \ldots < 1$, $a_n \to 0$, $b_n \to 1$. By the Lemma, we obtain a sequence (u_n) of positive $C^2[0,1]$ solutions which can be represented by Green function as

$$u_n(t) = (1-t)\int_0^t s p_n(s) u_n^{\lambda}(s)ds + t\int_t^1 (1-s)p_n(s)u_n^{\lambda}(s)ds.$$

We claim that (u_n) is uniformly bounded and equicontinuous. Indeed, by (3)

$$u_n(t) \le \int_0^1 s(1-s)p_n(s)u_n^{\lambda}(s)ds \le \left(|u_n|_\infty\right)^{\lambda}\int_0^1 s(1-s)p(s)ds,$$

i.e. $|u_n|_\infty \le M^{1/(1-\lambda)}$ where $M = \int_0^1 s(1-s)p(s)ds$. Also, for $0 \le t_1 < t_2 \le 1$ we have

$$|u_n(t_1) - u_n(t_2)| \le \left| \int_0^{t_1} s(1-t_1)p_n(s)u_n^{\lambda}(s)ds - \int_0^{t_2} s(1-t_2)p_n(s)u_n^{\lambda}(s)ds \right|$$
$$+ \left| \int_{t_1}^1 t_1(1-s)p_n(s)u_n^{\lambda}(s)ds - \int_{t_2}^1 t_2(1-s)p_n(s)u_n^{\lambda}(s)ds \right|$$
$$\le \int_0^{t_1} s(t_2 - t_1)p(s)u_n^{\lambda}(s)ds + \int_{t_1}^{t_2} s(1-t_2)p(s)u_n^{\lambda}(s)ds$$
$$+ \int_{t_1}^{t_2} t_1(1-s)p(s)u_n^{\lambda}(s)ds + \int_{t_2}^1 (t_2 - t_1)(1-s)p(s)u_n^{\lambda}(s)ds,$$

and since

$$t_2 - t_1 \le 1 - t_1 \le 1 - s \quad \text{if} \quad 0 \le s \le t_1,$$
$$1 - t_2 \le 1 - s \quad \text{if} \quad t_1 \le s \le t_2,$$
$$t_1 \le s \quad \text{if} \quad t_1 \le s \le t_2,$$
$$t_2 - t_1 \le t_2 \le s \quad \text{if} \quad t_2 \le s \le 1,$$

and $|u_n|_\infty \le M^{1/(1-\lambda)}$ Lebesgue Theorem implies $|u_n(t_1) - u_n(t_2)| \to 0$ as $t_2 \to t_1$ uniformly with respect to n. Hence by Arzela–Ascoli Theorem the sequence $(u_n(t))$

possesses a uniformly convergent subsequence. Finally, the limit function $u \in C[0,1] \cap C^2(0,1)$ is a solution of (1), (2).

Unfortunately, it is not clear that $u(t)$ is a nontrivial function. In order to overcome that problem, a modification of the nonlinearity is needed on each step in construction of the approximated problems (P_n). Thus we obtain an increasing sequence of positive solutions (u_n) which is uniformly bounded and equicontinuous by the same reasons as above. ■

THIRD PROOF BY FIXED POINT ARGUMENTS

Here we give another proof of the Main Theorem based on theorem of Rothe.

THEOREM (ROTHE, [K]**)** *Let T be a positive compact operator in Banach space E with a cone K. Then for every $r > 0$ T has at least one eigenfunction $v_r \in K$, $\|v_r\| = r$, corresponding to the positive eigenvalue μ_r, if*

$$\inf_{v \in K, \|v\| = r} \|Tv\| > 0 .$$

THIRD PROOF OF THE MAIN THEOREM Consider the operator T defined by

$$Tu = (1-t)\int_0^t sp(s)u^{\lambda}(s)ds + t\int_t^1 (1-s)p(s)u^{\lambda}(s)ds$$

on the cone

$$K = \{ u \in C[0,1] : u \text{ is concave, and } u(0) = u(1) = 0 \}.$$

We claim

1. $T : K \to K$

Indeed, $Tu \in C^2(0,1)$ and as in the first proof of the Main Theorem there exist the limits $Tu(0^+) = Tu(1^-) = 0$. The concavity of Tu follows by $(Tu)'' = -p(t)u^{\lambda} \le 0$.

2. T is positive.

For any $u \in K,\ u \ne 0$, by the concavity we get $u(t) > 0,\ t \in (0,1)$. Hence $Tu > 0$.

3. T is compact.

Indeed, for $u \in K$ with supremum norm $|u|_{\infty} = M$ by (3) we have

$$|Tu|_{\infty} \le M^{\lambda} \int_0^1 s(1-s)p(s)ds < \infty .$$

Moreover, it can be shown as in the second proof of the Main Theorem that $|Tu(t_1) - Tu(t_2)| \to 0$ as $t_2 \to t_1$. Thus by Arzela–Ascoli theorem T is compact on K.

4. $\inf_{u \in K, |u|_{\infty} = 1} |Tu|_{\infty} > 0$.

We denote $\varphi(t) = 1/2 - |t - 1/2| \in K$. For any concave function $u \in K,\ |u|_{\infty} = 1$ we have $u(t) \ge \varphi(t)$, $t \in [0,1]$. Then $\inf_{u \in K, |u|_{\infty} = 1} |Tu|_{\infty} \ge T\varphi > 0$, and the assertion follows.

Consequently, by Theorem of Rothe there are $v \in K$, $|v|_\infty = 1$ and $\mu > 0$ such that $Tv = \mu v$. Then $u \in C[0,1] \cap C^2(0,1)$ defined by

$$u = \mu^{\frac{1}{1-\lambda}} v$$

is desired positive solution of (1), (2). The proof is complete.■

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ТРИ ПОДХОДА КЪМ УРАВНЕНИЕТО НА ЕМДЕН – ФАУЛЪР В СУБЛИНЕЙНИЯ СЛУЧАЙ

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Резюме: *В тази статия са представени три различни доказателства за съществуване на положителни решения на сингулярна гранична задача, свързана с уравнението на Емден – Фаулър. Демонстрирано е разнообразие от съвременни методи за изследване на диференциални задачи, като методи на горно и долно решение, вариационни методи и теореми за неподвижни точки на положителни оператори в Банахови пространства с положителен конус.*

Ключови думи: *сингулярни гранични задачи, класически решения, Теорема на Роте, минимизация на функционали.*

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