FOURTH-ORDER DIFFERENCE SCHEMES FOR TWO POINTS BOUNDARY VALUE PROBLEM OF A FOURTH ORDER DEGENERATE DIFFERENTIAL EQUATION

Ivanka Angelova

Angel Kanchev University of Ruse

Abstract: *High-order finite difference approximations to the solutions of a fourth order degenerate differential equation are constructed. The integral identity method is implemented. Error analysis is given along numerical experiments.*

Key words: *Fourth order degenerate differential equation, two points boundary value problem, integral identity, finite difference schemes.*

INTRODUCTION

In this paper we propose a high-accurate numerical method for fourth order degenerate ordinary differential equations (ODEs). Such equations have many applications in physics, mechanics, engineering, etc.

Consider the following two-points boundary value problem of the fourth order:

$$\mathrm{L}u := (x^{\alpha}u'')'' + a(x)u = f(x) \quad on \quad (0,1),\ 0 \le \alpha < 1, \tag{1}$$

with boundary conditions

$$u(0) = u'(0) = 0,\ u(1) = u'(1) = 0. \tag{2}$$

We assume that $0 < a(x) < a_0 = const$. It is proved in [3] that the operator $\mathrm{L}u$ is positively definite on the space $L_2(0,1)$. Therefore, the problem (1)-(2) has an unique generalized solution $u \in W_{\alpha}^{2}$, which is a closure of the set of functions $C_0^{\infty}(0,1)$ in the norm

$$|u|_{W_{\alpha}^{2}} = \left(\int_0^1 (x^{\alpha} |u''(x)|^2 + |u(x)|^2) dx \right)^{1/2}.$$

The generalized solution is based on the integral identity.

In the paper [4] the author investigates the Neuman problem for equation (1) and in the next paper [5] he develops FEM approximations. The paper [6] implements the augmented immersed interface method for 1D and 2D problems.

The layout of the present paper is as follows. In the next section we introduce an integral operator that is a basic tool for construction of the fourth-order difference schemes. In the third section the difference schemes are derived. The fourth section is devoted to numerical experiments and finally, some conclusions are presented.

INTEGRAL OPERATOR

Let introduce on $[0,1]$ the set of mesh points: $x_0 = 0$, $x_i = ih$, $i = 1, \ldots, N$, $x_N = 1$, $h = 1/N$.

We introduce the left, right and central integral operators, respectively:

$$I_i^l(f) = \frac{2}{h^2} \int_{x_{i-1}}^{x_i} dx \int_x^{x_i} f(t)dt. \tag{3}$$

$$I_i^r(f) = \frac{2}{h^2} \int_{x_i}^{x_{i+1}} dx \int_{x_i}^{x} f(t)dt. \tag{4}$$

$$I_i(f)=\frac{1}{2}\left(I_i^l(f)+I_i^r(f)\right). \quad (5)$$

From (5) by change of order of integration we have

$$I_i(f)=\frac{1}{h^2}\left(\int_{x_{i-1}}^{x_i} f(x)(x-x_{i-1})dx+\int_{x_i}^{x_{i+1}} f(x)(x_{i+1}-x)dx\right). \quad (6)$$

Using (6) for (3) and (4) we get

$$I_i^l(f)=f_i-\frac{h}{3}f_i'+\frac{h^2}{12}f_i''-\frac{h^3}{60}f_i'''+O(h^4);\ I_i^l(f)=\frac{2f_i+f_{i-1}}{3}+O(h^2). \quad (7)$$

$$I_i^r(f)=f_i+\frac{h}{3}f_i'+\frac{h^2}{12}f_i''+\frac{h^3}{60}f_i'''+O(h^4);\ I_i^r(f)=\frac{2f_i+f_{i+1}}{3}+O(h^2). \quad (8)$$

$$I_i(f)=f_i+\frac{h^2}{12}f_i''+O(h^4)=\frac{f_{i-1}+10f_i+f_{i+1}}{12}+O(h^4). \quad (9)$$

It is easily to check that:

$$I_i^l(0)=I_i^r(0)=I_i(0)=0;\ I_i^l(1)=I_i^r(1)=I_i(1)=1; \quad (10)$$

$$I_i^l(u'')=\frac{2}{h}\left(u_i'-u_{\bar{x}_i}\right);\ I_i^r(u'')=\frac{2}{h}\left(u_{x_i}-u_i'\right);\ I_i(u'')=u_{\bar{x}x_i}=\frac{1}{h^2}\left(u_{i+1}-2u_i+u_{i-1}\right), \quad (11)$$

where $u_{\bar{x}_i}=\frac{u_i-u_{i-1}}{h}$; $u_{x_i}=\frac{u_{i+1}-u_i}{h}$.

DERIVATION OF DIFFERENCE SCHEMES

We rewrite the problem (1) - (2) in the mixed form:

$$\begin{aligned}&-x^\alpha u''-v=0,\\&-v''+a(x)u=f(x),\\&u(0)=u'(0)=0,\ u(1)=u'(1)=0.\end{aligned} \quad (12)$$

Therefore in view of (10)

$$I_i(-v''+a(x)u-f(x))=0;\ \ I_i\left(-u''-\frac{v}{x^\alpha}\right)=0;$$

Next, in view of (9) and (11)

$$-v_{\bar{x}x_i}+\frac{a_{i-1}u_{i-1}+10a_iu_i+a_{i+1}u_{i+1}}{12}=\frac{f_{i-1}+10f_i+f_{i+1}}{12}+O(h^4). \quad (13)$$

Let U and V be numerical approximations of the exact solutions u and v in (12). Then

$$-V_{\bar{x}x_i}+\frac{a_{i-1}U_{i-1}+10a_iU_i+a_{i+1}U_{i+1}}{12}=\frac{f_{i-1}+10f_i+f_{i+1}}{12}.$$

If $\alpha=0$ then it follows from (9) and (11)

$$-u_{\bar{x}x_i}-\frac{v_{i-1}+10v_i+v_{i+1}}{12}=O(h^4). \quad (14)$$

Then for the mesh functions U and V one has

$$-U_{\bar{x}x_i}-\frac{V_{i-1}+10V_i+V_{i+1}}{12}=0.$$

Using (10) and (11), we have $I_i^l(-v''+a(x)u-f(x))=0$, $I_i^r(-v''+a(x)u-f(x))=0$.

Then we find $v'(x_i-0)=v_{\bar{x}_i}+\frac{h}{2}I_i^l(a(x)u-f(x))$, $v'(x_i+0)=v_{x_i}-\frac{h}{2}I_i^r(a(x)u-f(x))$.

If $\alpha\in(0,1)$ then using the equation in (12) we find

$$-u_{\bar{x}x_i} - I_i\left(\frac{v}{x^\alpha}\right) = 0. \qquad (15)$$

Next, using (6) we get

$$I_i\left(\frac{v}{x^\alpha}\right) = \frac{1}{h^2}\left(\int_{x_{i-1}}^{x_i} v(x)\frac{x-x_{i-1}}{x^\alpha}dx + \int_{x_i}^{x_{i+1}} v(x)\frac{x_{i+1}-x}{x^\alpha}dx\right);$$

For $i=1,\dots,N-1$ we introduce the functions

$$\xi_i^{(2)}(x) = \begin{cases} \int\limits_{x_{i-1}}^{x} \frac{t-x_{i-1}}{t^\alpha}dt, x\in(x_{i-1},x_i), \\ \int\limits_{x}^{x_{i+1}} \frac{x_{i+1}-t}{t^\alpha}dt, x\in(x_i,x_{i+1}), \\ 0, x\notin[x_{i-1},x_{i+1}]. \end{cases}$$

For $m=2,3,4$ let introduce

$$\xi_i^{(m+1)}(x) = \begin{cases} \int\limits_{x_{i-1}}^{x} \xi_i^{(m)}(t)\,dt, x\in(x_{i-1},x_i), \\ \int\limits_{x}^{x_{i+1}} \xi_i^{(m)}(t)\,dt, x\in(x_i,x_{i+1}), \\ 0, x\notin[x_{i-1},x_{i+1}]. \end{cases}$$

Next, we denote: $\xi_i^{(m)}(x_i\pm 0) = \xi_{i\pm}^{(m)}$. Then, integrating by parts, we have:

$$I_i^l\left(\frac{v}{x^\alpha}\right) = \frac{2}{h^2}\left(\xi_{i-}^{(2)}v_i - \xi_{i-}^{(3)}\left(v_{\bar{x}_i} + \frac{h}{2}I_i^l(au-f)\right) + \int_{x_{i-1}}^{x_i}(au-f)\xi_i^{(3)}(x)dx\right).$$

$$I_i^r\left(\frac{v}{x^\alpha}\right) = \frac{2}{h^2}\left(\xi_{i+}^{(2)}v_i + \xi_{i+}^{(3)}\left(v_{x_i} - \frac{h}{2}I_i^r(au-f)\right) + \int_{x_i}^{x_{i+1}}(au-f)\xi_i^{(3)}(x)dx\right).$$

$$\int_{x_{i-1}}^{x_i} f(x)\xi_i^{(3)}(x)dx = \left(\xi_{i-}^{(4)} - \frac{\xi_{i-}^{(5)}}{h}\right)f_i + \frac{\xi_{i-}^{(5)}}{h}f_{i-1} + O(h^{6-\alpha}).$$

$$\int_{x_i}^{x_{i+1}} f(x)\xi_i^{(3)}(x)dx = \left(\xi_{i+}^{(4)} - \frac{\xi_{i+}^{(5)}}{h}\right)f_i + \frac{\xi_{i+}^{(5)}}{h}f_{i+1} + O(h^{6-\alpha}).$$

Now, coming back to (15), we obtain the approximation of the ODE system of equations in (12)

$$-u_{\bar{x}x_i} - \frac{\xi_{i-}^{(3)}}{h^3}v_{i-1} - \left(\frac{\xi_{i-}^{(2)}+\xi_{i+}^{(2)}}{h^2} - \frac{\xi_{i-}^{(3)}+\xi_{i+}^{(3)}}{h^3}\right)v_i - \frac{\xi_{i+}^{(3)}}{h^3}v_{i+1}$$

$$+\left(\frac{\xi_{i-}^{(3)}}{6h}-\frac{\xi_{i-}^{(5)}}{h^3}\right)(au-f)_{i-1}+\left(\frac{\xi_{i+}^{(3)}}{6h}-\frac{\xi_{i+}^{(5)}}{h^3}\right)(au-f)_{i+1} \tag{16}$$

$$+\left(\frac{\xi_{i-}^{(3)}+\xi_{i+}^{(3)}}{3h}-\frac{\xi_{i-}^{(4)}+\xi_{i+}^{(4)}}{h^2}+\frac{\xi_{i-}^{(5)}+\xi_{i+}^{(5)}}{h^3}\right)(au-f)_i=O(h^{4-\alpha}).$$

Neglecting the term $O(h^{4-\alpha})$ and replacing u_i by U_i, v_i by V_i, we obtain the difference scheme:

$$-U_{\bar{x}x,i}-\frac{\xi_{i-}^{(3)}}{h^3}V_{i-1}-\left(\frac{\xi_{i-}^{(2)}+\xi_{i+}^{(2)}}{h^2}-\frac{\xi_{i-}^{(3)}+\xi_{i+}^{(3)}}{h^3}\right)V_i-\frac{\xi_{i+}^{(3)}}{h^3}V_{i+1}$$

$$+\left(\frac{\xi_{i-}^{(3)}}{6h}-\frac{\xi_{i-}^{(5)}}{h^3}\right)(aU-f)_{i-1}+\left(\frac{\xi_{i+}^{(3)}}{6h}-\frac{\xi_{i+}^{(5)}}{h^3}\right)(aU-f)_{i+1}$$

$$+\left(\frac{\xi_{i-}^{(3)}+\xi_{i+}^{(3)}}{3h}-\frac{\xi_{i-}^{(4)}+\xi_{i+}^{(4)}}{h^2}+\frac{\xi_{i-}^{(5)}+\xi_{i+}^{(5)}}{h^3}\right)(aU-f)_i=0.$$

Further, we approximate the boundary conditions (2).

From $I_0^r(-v''+a(x)u-f(x))=0$ it follows $v'(0)=v_{x_0}-\frac{h}{2}I_0^r(a(x)u)+\frac{h}{2}I_0^r(f(x))$.

From $I_N^l(-v''+a(x)u-f(x))=0$ it follows $v'(1)=v_{\bar{x}_N}+\frac{h}{2}I_N^l(a(x)u)-\frac{h}{2}I_N^l(f(x))$.

From (2) we get $I_0^r(a(x)u)=O(h^2)$, $I_N^l(a(x)u)=O(h^2)$.

Equation (11) gives $I_0^r(u'')=\frac{2}{h^2}u_1$, $I_N^l(u'')=\frac{2}{h^2}u_{N-1}$;

$\int_0^{x_1}a(x)u\,\xi_i^{(3)}(x)dx=O(h^{6-\alpha})$; $\int_{x_{N-1}}^1 a(x)u\,\xi_i^{(3)}(x)dx=O(h^{6-\alpha})$.

$$I_0^r\left(\frac{v}{x^\alpha}\right)=\frac{2}{h^2}\left(\xi_{0+}^{(2)}v_0+\xi_{0+}^{(3)}\left(v_{x_0}-\frac{h}{2}I_0^r(au-f)\right)+\int_0^{x_1}(au-f)\xi_0^{(3)}(x)dx\right).$$

$$I_N^l\left(\frac{v}{x^\alpha}\right)=\frac{2}{h^2}\left(\xi_{N-}^{(2)}v_N-\xi_{N-}^{(3)}\left(v_{\bar{x}N}+\frac{h}{2}I_N^l(au-f)\right)+\int_{x_{N-1}}^1(au-f)\xi_N^{(3)}(x)dx\right).$$

From $I_0^r(-u''-\frac{v}{x^\alpha})=0$ it follows that

$$-\frac{2}{h^2}u_1-\frac{2}{h^2}\left(\xi_{0+}^{(2)}v_0+\xi_{0+}^{(3)}\left(v_{x_0}+\frac{h}{2}I_0^r(f(x))-\int_0^{x_1}f(x)\xi_0^{(3)}(x)dx\right)\right)=O(h^{4-\alpha}).$$

From $I_N^l(-u''-\frac{v}{x^\alpha})=0$ we get

$$-\frac{2}{h^2}u_{N-1}-\frac{2}{h^2}\left(\xi_{N-}^{(2)}v_N-\xi_{N-}^{(3)}\left(v_{\bar{x}N}-\frac{h}{2}I_N^l(f)\right)-\int_{x_{N-1}}^1 f(x)\xi_N^{(3)}(x)dx\right)=O(h^{4-\alpha}).$$

We obtain the following approximations of the boundary conditions (2):

$$\frac{1}{h^2}U_1+\left(\frac{\xi_{0+}^{(2)}}{h^2}-\frac{\xi_{0+}^{(3)}}{h^3}\right)V_0+\frac{\xi_{0+}^{(3)}}{h^3}V_1=-\left(\frac{\xi_{0+}^{(3)}}{3h}-\frac{\xi_{0+}^{(4)}}{h^2}+\frac{\xi_{0+}^{(5)}}{h^3}\right)f_0-\left(\frac{\xi_{0+}^{(3)}}{6h}-\frac{\xi_{0+}^{(5)}}{h^3}\right)f_1.$$

$$\frac{1}{h^2}U_{N-1}+\left(\frac{\xi_{N-}^{(2)}}{h^2}-\frac{\xi_{N-}^{(3)}}{h^3}\right)V_N+\frac{\xi_{N-}^{(3)}}{h^3}V_{N-1}=-\left(\frac{\xi_{N-}^{(3)}}{3h}-\frac{\xi_{N-}^{(4)}}{h^2}+\frac{\xi_{N-}^{(5)}}{h^3}\right)f_N-\left(\frac{\xi_{N-}^{(3)}}{6h}-\frac{\xi_{N-}^{(5)}}{h^3}\right)f_{N-1}.$$

Finally, to complete the difference scheme approximation of problem (1), (2) we add the boundary coditions: $U_0 = 0$, $U_N = 0$.

NUMERICAL EXPERIMENTS

In this section are presented samples of numerical experiments illustrating the accuracy of the schemes derived in the previous section.

Example 1. We consider the BVP (1)-(2), where $a(x) = 1 + e^x$ with an exact solution:

$$u(x) = x^{2-\alpha}\left(1 - \sin \pi x/2\right)^2 .$$

We define the strong mesh norm and rate of convergence, respectively for $z_u(x_i) = u(x_i) - U_i; \quad z_v(x_i) = v(x_i) - V_i$.

$$\left\|z_u^N\right\|_\infty = \max_{0\le i\le N} |z_u(x_i)|, \quad \left\|z_v^N\right\|_\infty = \max_{0\le i\le N} |z_v(x_i)|, \ p_u = \log_2 \frac{\left\|z_u^N\right\|_\infty}{\left\|z_u^{2N}\right\|_\infty}, \ p_v = \log_2 \frac{\left\|z_v^N\right\|_\infty}{\left\|z_v^{2N}\right\|_\infty}.$$

The approximations were computed on uniform meshes with $N = 8,16,\ldots,256$. For the strong norm in the tables below the subscript ∞ is omitted.

Table 1: *Example* 1: *the solution u and v* for $\alpha = 0$.

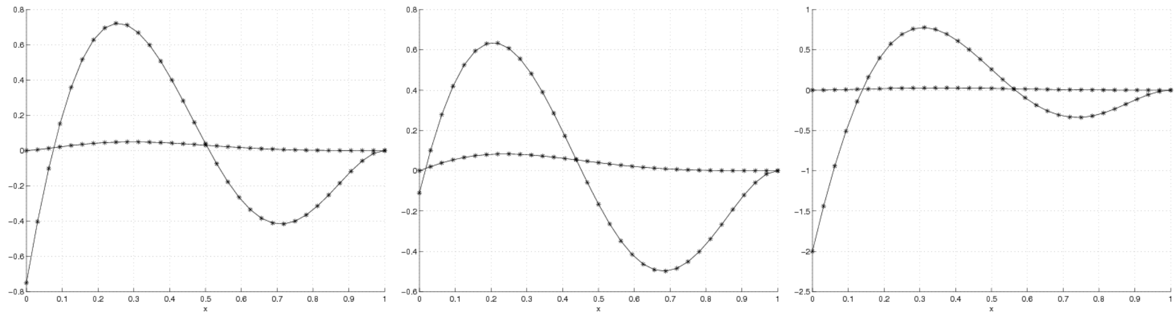
n/N	8	16	32	64	128	256	512
$\| z_u \|$	5.144E-5	2.986E-6	1.860E-7	1.175E-8	7.405E-10	4.651E-11	2.933E-12
p_u	4.1066	4.0046	3.9850	3.9878	3.9929	3.9869	
$\| z_v \|$	1.960E-3	1.648E-4	1.148E-5	7.524E-7	4.806E-8	3.036E-9	1.910E-10
p_v	3.5722	3.8426	3.9321	3.9684	3.9849	3.9906	

Table 2: *Example* 1: *the solution u and v* for $\alpha = 0.5$.

n/N	8	16	32	64	128	256	512
$\| z_u \|$	2.560E-5	1.171E-6	7.685E-8	5.330E-9	3.560E-10	2.054E-11	1.779E-12
p_u	4.4501	3.9300	3.8497	3.9041	4.1158	3.5286	
$\| z_v \|$	2.210E-3	1.406E-4	9.058E-6	5.806E-7	3.693E-8	2.350E-9	1.999E-10
p_v	3.9743	3.9565	3.9637	3.9745	3.9744	3.5550	

Table 3: *Example* 1: *the solution u and v* for $\alpha = 0.9$.

n/N	8	16	32	64	128	256	512
$\| z_u \|$	2.076E-4	1.249E-5	7.427E-7	4.490E-8	2.803E-9	2.743E-10	3.729E-10
p_u	4.0554	4.0712	4.0480	4.0020	3.3529	-0.4430	
$\| z_v \|$	1.302E-3	8.304E-5	5.298E-6	3.361E-7	2.130E-8	2.498E-9	4.754E-9
p_v	3.9712	3.9703	3.9785	3.9797	3.0923	-0.9285	



1) 2) 3)

Figure 1: Approximate and exact solution u and v for Example 1 and $N = 32$:
1) $\alpha = 0$; 2) $\alpha = 0.5$; 3) $\alpha = 0.9$.

CONCLUSIONS

In this paper we generalized the integral identity for a fourth order degenerate ODE, first proposed for second order elliptic problems in [1, 2]. We have come to the following conclusions:

• The proposed difference method is easy to implement compared with other methods.

• This integral identity method have less regularity requirements.

• The method can be applied to problems of elasticity.

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CONTACT ADDRESS

Pr. Assist.Ivanka Angelova, PhD
Department of Mathematics
Faculty of Natural Sciences and Education
Angel Kanchev University of Ruse
8, Studentska Str.
7017 Ruse, Bulgaria
Phone:(++359 82)888 587
E-mail: iangelova@uni-ruse.bg

ДИФЕРЕНЧНИ СХЕМИ ОТ ЧЕТВЪРТИ РЕД НА ТОЧНОСТ ЗА ИЗРОДЕНИ ОБИКНОВЕНИ ДИФЕРЕНЦИАЛНИ УРАВНЕНИЯ ОТ ЧЕТВЪРТИ РЕД С ДВУТОЧКОВИ ГРАНИЧНИ УСЛОВИЯ

Иванка Ангелова

Русенски университет "Ангел Кънчев"

Резюме: *Изведена е схема от четвърти ред на точност за изродени диференциални уравнения от четвърти ред. За извеждането на диференчните схеми използваме интегрални тъждества. Направени са числени експерименти за различни стойности на параметъра* α *.*

Ключови думи: *Изродени диференциални уравнения от четвърти ред, двуточкови гранични условия, интегрални тъждества, диференчни схеми.*

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