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"MATHEMATICS, INFORMATICS AND PHYSICS"

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"MATHEMATICS, INFORMATICS AND PHYSICS"

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MATHEMATICAL MODELS OF INTERFACE PROBLEMS FOR STEADY-UNSTEADY HEAT CONDUCTION

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***Abstract**: We study mathematical models describing non-stationary heat conduction in two bodies separated by steady conductor (isolation). These problems are related to parabolic equations with discontinuous coefficients and concentrated sources. As a result across the interfaces contact (jump) conditions arise. It is proved that this problem can be reduced to a variational problem. An asymptotic analysis of the interface problem is derived for the case when the thickness of the isolation tends to zero. As a result a new parabolic interface problem with non-ideal contact conditions is derived.*

***Keywords:** Heat conduction, interface, contact conditions, elliptic and parabolic equations, asymptotical analysis*

INTRODUCTION

There are two main reasons for coupling different models in different regions: the first are problems where the physics is different in different regions [2, 3, 7], and hence different models need to be used, for example in steady-unsteady heat conduction coupling. The second are problems where one is in principle interested in the full physical model, but the full model is too expensive computationally over entire region, and hence one would like to use a simple model in parts of the region, and full one only where it is essential to capture the physical phenomena [3, 5, 9]. Here we will discuss models that concern the two cases.

We consider a general problem where the domain consist of the following three parts: unsteady conductor - steady conductor - unsteady conductor. A survey on mathematical aspects of coupled conduction-radiation energy transfer problems is given in [7].

The stability of solutions to interface problems is studied in [4]. Elliptic and parabolic interface problems on disjoint domains are studied in [4, 9].

One is also interested in efficient algorithms to solve the coupled problems. One of the most effective method is the so called immersed interface method (IIM) [6]. This method uses uniform meshes and has second order of local approximation.

We want to emphasize that the purpose of this paper is not only to model a realistic application, but also to study theoretically a prototype linear initial boundary-value problems.

The paper is organized as follows. Our model is presented in Section 2. We prove that this problem can be reduces to variational problem, Theorem 1. In Section 3 we investigate the case when the thickness of the isolation tends to zero. The result is an interface problem with *non-ideal contact* conditions.

GENERAL MATHEMATICAL MODEL

Let consider the steady state heat conduction problem, Fig.1: differential equations

$$\frac{\partial u_i}{\partial t} - \frac{\partial}{\partial x}\left(p_i(x,y), \frac{\partial u_i}{\partial x} \right) - \frac{\partial}{\partial y}\left(q_i(x,y), \frac{\partial u_i}{\partial y} \right) = f_i(x,y,t), \qquad (1)$$

$$(x,y) \in \Omega_i \equiv (a_i, b_i) \times (c,d), \ 0 < t < T, \ i = 1,2; \ -\infty < a_1 < b_1 < a_2 < b_2 < +\infty;$$

$$-k\left(\frac{\partial^2 u_0}{\partial x^2}+\frac{\partial^2 u_0}{\partial y^2}\right) = f_0(x,y,t), \ \ (x,y)\in\Omega_0 \equiv (b_1,a_2)\times(c,d), \ \ 0<t<T; \quad (2)$$

$$[u]_{x=b_1} = u_0(b_1,y,t)-u_1(b_1,y,t) = 0, \ \left(k\frac{\partial u_0}{\partial x}\right)_{x=b_1} - \left(p_1\frac{\partial u_1}{\partial x}\right)_{x=b_1} = 0, \quad (3)$$

$$[u]_{x=a_2} = u_2(a_2,y,t)-u_0(a_2,y,t) = 0, \ \left(p_2\frac{\partial u_2}{\partial x}\right)_{x=a_2} - \left(k\frac{\partial u_0}{\partial x}\right)_{x=a_2} = 0; \quad (4)$$

zero-Dirichlet boundary conditions:

$$u_1(a_1,y,t) = 0, \quad u_2(b_2,y,t) = 0, \quad y\in(c,d), \quad (5)$$
$$u_1(x,c,t)=u_1(x,d,t)=0, \ x\in(a_1,b_1); \ u_2(x,c,t)=u_2(x,d,t)=0, \ x\in(a_2,b_2). \quad (6)$$

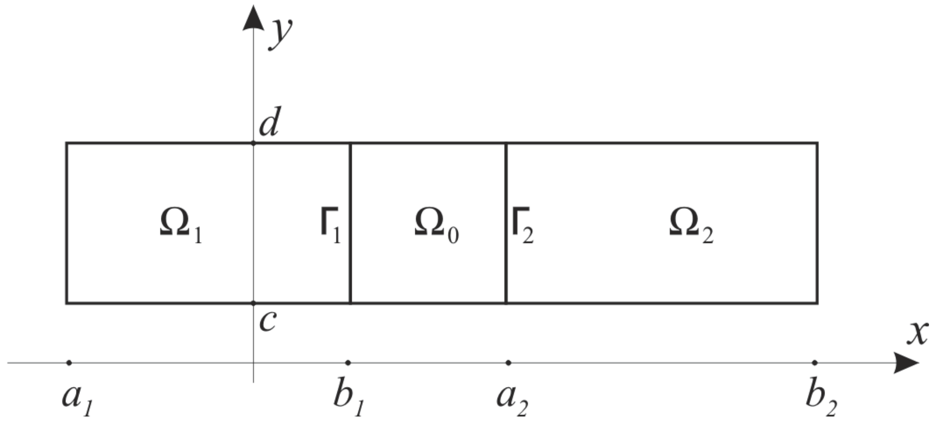

Figure 1: The geometry of bodies Ω_1, Ω_2 conductivities with thin isolation Ω_0.

Let us note that the heat transfers between conductivities and isolation across the interfaces

$$\Gamma_1 = \{(b_1,y): y\in(c,d)\}, \quad \Gamma_2 = \{(a_2,y): y\in(c,d)\}$$

is realized according to the *ideal contact* conditions (3) , (4).

Finally, in order to complete the initial boundary value problem we pose initial conditions

$$u_i(x,y,0) = u_{i0}(x,y), \ i = 1,0,2. \quad (7)$$

Throughout the paper we assume for $i = 1,2$ the usual regularity conditions

$$p_i(x,y)>0, q_i(x,y)>0, \ p_i, q_i \in C^1(\Omega_i), k = const.>0, \ f_i(x,y,t)\in C(\Omega_i)\times C(0,T). \quad (8)$$

Then we can prove that the solution of the interface problem (1)-(8)

$$u \equiv (u_1,u_0,u_2)\in \left(C^2(\Omega_1\cup\Omega_0\cup\Omega_2)\cap C(\Omega)\right)\times C^1(0,T).$$

Let us introduce the piecewise continuous coefficients

$$p(x,y) = \begin{cases} p_1(x,y) & ,(x,y)\in\Omega_1, \\ k & ,(x,y)\in\Omega_0, \\ p_2(x,y) & ,(x,y)\in\Omega_2; \end{cases} \quad q(x,y) = \begin{cases} q_1(x,y) & ,(x,y)\in\Omega_1, \\ k & ,(x,y)\in\Omega_0, \\ q_2(x,y) & ,(x,y)\in\Omega_2; \end{cases}$$

and the function

$$F(x,y,t)=\begin{cases} f_1(x,y,t) & ,(x,y)\in\Omega_1,\\ f_0(x,y,t) & ,(x,y)\in\Omega_0,\\ f_2(x,y,t) & ,(x,y)\in\Omega_2.\end{cases}$$

We can prove the following assertion.

THEOREM 1. Let (7), (8) hold. Then the solution $u\in\left(C^2(\Omega_1\cup\Omega_0\cup\Omega_1)\cap C(\Omega)\right)\times C^1(0,T)$ of the interface problem (1)-(8) is also the solution of the variational problem: find $u\in H^1(\Omega)\times C^1(0,T)$ with boundary conditions (5), (6), which satisfies the following integral identity

$$\iint_{\Omega_1} v_1(x,y)\frac{\partial u_1}{\partial t}dxdy+\iint_{\Omega_2} v_2(x,y)\frac{\partial u_2}{\partial t}dxdy$$
$$+\iint_{\Omega}\left(p(x,y)\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+q(x,y)\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)dxdy=\iint_{\Omega} v(x,y)F(x,y,t)dxdy,\ \forall\, v\in \overset{\circ}{H}{}^1(\Omega).$$

Proof. Let us multiply the both sides of (1), $i=1,2$ and (2) by $v\equiv(v_1(x,y),\ v_0(x,y),\ v_2(x,y))$ and integrate on Ω_1,Ω_2 and Ω_0 separately. Then, applying integration by parts, we obtain

$$\iint_{\Omega_1} v_1(x,y)\frac{\partial u_1}{\partial t}dxdy+\iint_{\Omega_1}\left(p_1(x,y)\frac{\partial u_1}{\partial x}\frac{\partial v_1}{\partial x}+q_1(x,y)\frac{\partial u_1}{\partial y}\frac{\partial v_1}{\partial y}\right)dxdy$$
$$-\int_c^d\left(v_1 p_1\frac{\partial u_1}{\partial x}\right)(b_1,y,t)dy=\iint_{\Omega_1} v_1(x,y)\,f_1(x,y,t)dxdy.$$

$$k\iint_{\Omega_0}\left(\frac{\partial u_0}{\partial x}\frac{\partial v_0}{\partial x}+\frac{\partial u_0}{\partial y}\frac{\partial v_0}{\partial y}\right)dxdy+k\int_c^d\left[\left(v_0\frac{\partial u_0}{\partial x}\right)(b_1,y,t)-\left(v_0\frac{\partial u_0}{\partial x}\right)(a_2,y,t)\right]dy$$
$$=\iint_{\Omega_0} v_0(x,y)\,f_0(x,y,t)dxdy.$$

$$\iint_{\Omega_2} v_2(x,y)\frac{\partial u_2}{\partial t}dxdy+\iint_{\Omega_2}\left(p_2(x,y)\frac{\partial u_2}{\partial x}\frac{\partial v_2}{\partial x}+q_2(x,y)\frac{\partial u_2}{\partial y}\frac{\partial v_2}{\partial y}\right)dxdy$$
$$+\int_c^d\left(v_2 p_2\frac{\partial u_2}{\partial x}\right)(a_2,y,t)dxdy=\iint_{\Omega_2} v_2(x,y)\,f_2(x,y,t)dxdy.$$

Summing these identities, we get

$$\iint_{\Omega_1} v_1(x,y)\frac{\partial u_1}{\partial t}dxdy+\iint_{\Omega_1}\left(p_1(x,y)\frac{\partial u_1}{\partial x}\frac{\partial v_1}{\partial x}+q_1(x,y)\frac{\partial u_1}{\partial y}\frac{\partial v_1}{\partial y}\right)dxdy$$
$$+k\iint_{\Omega_0}\left(\frac{\partial u_0}{\partial x}\frac{\partial v_0}{\partial x}+\frac{\partial u_0}{\partial y}\frac{\partial v_0}{\partial y}\right)dxdy+k\int_c^d\left[\left(v_0\frac{\partial u_0}{\partial x}\right)(b_1,y,t)-\left(v_0\frac{\partial u_0}{\partial x}\right)(a_2,y,t)\right]dy$$
$$+\iint_{\Omega_2} v_2(x,y)\frac{\partial u_2}{\partial t}dxdy+\iint_{\Omega_2}\left(p_2(x,y)\frac{\partial u_2}{\partial x}\frac{\partial v_2}{\partial x}+q_2(x,y)\frac{\partial u_2}{\partial y}\frac{\partial v_2}{\partial y}\right)dxdy$$

$$+\int_c^d\left[\left(v_2 p_2\frac{\partial u_2}{\partial x}\right)(a_2,y,t)-\left(v_1 p_1\frac{\partial u_1}{\partial x}\right)(b_1,y,t)\right]dxdy = \iint_\Omega v(x,y)F(x,y,t)dxdy.$$

for all $v=v(x,y)\in \overset{\circ}{H}{}^1(\Omega)$. Since $u_0(b_1,y,t)=u_1(b_1,y,t),\ u_2(a_2,y,t)=u_0(a_2,y,t)$ it is natural to require that the arbitrary function $v=(v_1,v_0,v_2)$ also satisfies these conditions. Using these conditions we obtain

$$\int_c^d\left[\left(v_2 p_2\frac{\partial u_2}{\partial x}\right)(a_2,y,t)-\left(v_1 p_1\frac{\partial u_1}{\partial x}\right)(b_1,y,t)\right.$$
$$\left.+k\left(v_0\frac{\partial u_0}{\partial x}\right)(b_1,y,t)-k\left(v_0\frac{\partial u_0}{\partial x}\right)(a_2,y,t)\right]dxdy$$
$$=\int_c^d\left[v_0\left(p_2\frac{\partial u_2}{\partial x}-k\frac{\partial u_0}{\partial x}\right)(a_2,y,t)-v_0\left(k\frac{\partial u_0}{\partial x}-p_1\frac{\partial u_1}{\partial x}\right)(b_1,y,t)\right]dy=0.$$

We obtain

$$\iint_{\Omega_1}v_1(x,y)\frac{\partial u_1}{\partial t}dxdy+\iint_{\Omega_2}v_2(x,y)\frac{\partial u_2}{\partial t}dxdy$$
$$+\iint_{\Omega_1}\left(p_1(x,y)\frac{\partial u_1}{\partial x}\frac{\partial v_1}{\partial x}+q_1(x,y)\frac{\partial u_1}{\partial y}\frac{\partial v_1}{\partial y}\right)dxdy$$
$$+k\iint_{\Omega_0}\left(\frac{\partial u_0}{\partial x}\frac{\partial v_0}{\partial x}+\frac{\partial u_0}{\partial y}\frac{\partial v_0}{\partial y}\right)dxdy$$
$$+\iint_{\Omega_2}\left(p_2(x,y)\frac{\partial u_2}{\partial x}\frac{\partial v_2}{\partial x}+q_2(x,y)\frac{\partial u_2}{\partial y}\frac{\partial v_2}{\partial y}\right)dxdy$$
$$=\iint_\Omega v(x,y)F(x,y,t)dxdy.$$

This integral identity with the Neumann transmission conditions (3), (4) completes the proof [8, 10].

This theorem shows the equivalency of the transmission problem (1)-(3) with *variational problem*, although the last one do not contain any transmission condition. This suggests a possibility of construction of such finite difference (or finite element) analogues of the interface problem, which has the similar structure. Specifically, the question is to construct homogeneous conservative difference schemes in the sense of Samarskii [8], that have the same form for all mesh points including ones on the interfaces $\Gamma_1,\ \Gamma_2$.

THE LIMIT CASE $\varepsilon\to 0$

We assume that the thickness $2\varepsilon=a_2-b_1$ of the isolation Ω_0 is small , $\varepsilon<<1$ and the conductivity $k>0$, see Fig.1.

Suppose that we wish to solve numerically the problem (1)-(6) by difference scheme. If the value of thickness $2\varepsilon=a_2-b_1$ of the steady isolation Ω_0 is less than the mesh size h along the direction Ox , that is $\varepsilon<h$, then the interface conditions (3), (4) cannot be approximated on this mesh. To derive a finite difference approximation of the interface problem (1)-(6), one need to derive an asymptotic analysis of this problem, when $\varepsilon\to 0$.

Let us note that singularly perturbed one-dimensional interface problems were studied in [1].

In order to simplify the exhibition we take $b_1 = -\varepsilon$, $a_2 = \varepsilon$.

THEOREM 2. The limit case, when $\varepsilon \to 0$, (i.e. $b_1 \to a_2$), of the transmission problem (1)-(6) with *ideal* contact conditions (3), (4) is the following transmission problem with *nonideal* conditions:

$$\frac{\partial u_i}{\partial t} - \frac{\partial}{\partial x}\left(p_i(x,y)\frac{\partial u_i}{\partial x}\right) - \frac{\partial}{\partial y}\left(q_i(x,y)\frac{\partial u_i}{\partial y}\right) = f_i(x,y,t),\ (x,y)\in\Omega_i,\ i=1,2;\ t\in(0,T],$$

$$\left(p_1(x,y)\frac{\partial u}{\partial x}\right)_{x=0^-} = \sigma[u]_{x=0} = \left(p_2(x,y),\frac{\partial u}{\partial x}\right)_{x=0^+},\ \sigma = const. > 0, \tag{9}$$

$$u(x,y,t) = 0,\ (x,y)\in\partial(\Omega_1\cup\Omega_2),\ t\in(0,T],$$

$$u(x,y,0) = u_0(x,y),\ (x,y)\in\Omega_1\cup\Omega_2.$$

Proof. Let us assume that $u = u(x,y,t)$ is the solution of the transmission problem (1)-(6). Integrating (2) on $x\in(-\varepsilon,\xi)$, where $\xi\in(-\varepsilon,\varepsilon)$ and $\varepsilon>0$ is an arbitrary small parameter to get

$$-k\int_{-\varepsilon}^{\xi}\left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2}\right)dx = \int_{-\varepsilon}^{\xi} f_0(x,y,t)dx,$$

$$k\frac{\partial u_0}{\partial x}(-\varepsilon,y,t) - k\frac{\partial u_0}{\partial x}(\xi,y,t) = \int_{-\varepsilon}^{\xi}\left(k\frac{\partial^2 u_0}{\partial y^2} + f_0(x,y,t)\right)dx.$$

Since the parameter $\xi\in(-\varepsilon,\varepsilon)$ is an arbitrary one , integrating both sides of the last identity with respect to this parameter on $(-\varepsilon,\varepsilon)$ and use (3) and (4), we obtain

$$2\varepsilon p_1(-\varepsilon,y)\frac{\partial u_1}{\partial x}(-\varepsilon,y,t) - k\left[u_2(\varepsilon,y,t) - u_1(-\varepsilon,y,t)\right]$$
$$= \int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\xi}\left(k\frac{\partial^2 u_0}{\partial y^2} + f_0(x,y,t)\right)dx\,d\xi.$$

Let us divide now both sides by $2\varepsilon \neq 0$. Then passing to the limit as $\varepsilon, k \to 0$ and requiring $\sigma = \dfrac{k}{2\varepsilon} = const$, we get

$$\left(p_1\frac{\partial u_1}{\partial x}\right)(0-,y,t) = \sigma[u_2(0+,y,t) - u_1(0-,y,t)],\ y\in(c,d),\ t\in(0,T],$$

where $\xi \to 0$ as $\varepsilon \to 0$. This condition can be written in the following form:

$$\sigma[u]_{x=0} = \left(p_1\frac{\partial u}{\partial x}\right)_{x=0-}. \tag{10}$$

Integrating (2) on $x\in(\xi,\varepsilon\}$ respectively, by the same way, we can obtain the second limit condition

$$\sigma[u]_{x=0} = \left(p_2\frac{\partial u}{\partial x}\right)_{x=0+}. \tag{11}$$

Conditions (10) - (11) imply the transmission conditions (9). This completes the proof.

Asymptotic analysis to a one-dimensional version of the problem (9) is given in [1].

CONCLUSIONS

In this paper using three rectangles as the simplest case of multi component model an analysis of a parabolic-elliptic interface problem is presented. This study can obviously be extended to the problems with more complicated geometry. We hope that this short paper will trigger some subsequent works on analytical and numerical analysis of partial differential equations with nonstandard transmission conditions.

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МАТЕМАТИЧЕСКИ МОДЕЛИ НА ИНТЕРФЕЙСНИ ЗАДАЧИ ЗА СТАЦИОНАРНА - НЕСТАЦИОНАРНА ТОПЛОПРОВОДИМОСТ

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Резюме: Ние изследваме математически модели, описващи нестационарна топлопроводност в две тела, разделени със стационарен изолатор. Тези задачи са свързани с параболични уравнения с прекъснати коефициенти и концентрирани източници. Като резултат над интерфейса, възникват контактни (скокови) условия. Доказано е, че тази задача допуска вариационна формулировка. Извършен е асимптотичен анализ в случая, когато дебелината на изолатора клони към нула. Като резултат възниква нова параболична задача с неидеални контактни условия.

Ключови думи: Стационарна и нестационарни топлопроводимост, интерфейс, елиптични и параболични уравнения, асимптотичен анализ.

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