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"MATHEMATICS, INFORMATICS AND PHYSICS"

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NUMERICAL SOLUTION OF THE TWO-PHASE STEFAN PROBLEM FOR SPHERE

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***Abstract**: We consider two-phase Stefan problem for spheres in cases of small diffusion or large Stefan number limit. Landau's transformation with the Gupta & Kumar [1] variable time-step method to solve numerically the problem is combined. A second order difference scheme with respect to space is derived and an algorithm for solution of the algebraic equations system is proposed. Shishkin meshes are used in the region of small diffusion. Numerical experiments show more accurate results in case of Shishkin mesh.*

***Keywords:** Two-phase Stefan problem for spheres, free boundary problems, difference scheme, Shishkin mesh*

INTRODUCTION

One of the most simple moving boundary problems to pose is the classical Stefan problem for the inward solidification of a spherical ice ball. Even in this idealized case there is no (known) exact solution, and the only way to obtain meaningful results is through numerical or approximate means. In this study, the full two-phase problem is considered, and in particular, the attention is given to large Stefan limit or small diffusion. By applying the method of matched asymptotic expansion the temperature in both two phases the authors show that the solid-melt interface $r = R(t)$ moves slowly and the two phase are weakly coupled for large Stefan number ($\beta >> 1$). The singular region of small diffusion is considered.

The paper [9] is concerned with modelling the melting process of a nanoscaled sphere or cylinder and the resulting boundary value problem takes the form below.

Numerical analysis of heat and mass transfer with moving interface boundaries between two or more subdomains often bring us to diffraction boundary value problems. In the case of the presence of concentrated sources and small diffusion coefficient it is necessary to develop special numerical methods whose errors depend rather weakly on the parameter ε. The behavior of the solutions is very complicated in the case of moving concentrated sources [2, 6, 7, 8].

The purpose of the present study is using the Landau transformation to transform the two-phase Stefan problem for sphere into an interface problem. The left parabolic problem is defined on a rectangle and the right one is an one-phase Stefan problem while the interface is a segment parallel to axe Ot. We use layer adapted meshes (Shishkin's) [4, 6], see Fig.2 for the left problem in case of small diffusion. On the base of the Gupta & Kumar [1] variable time-step method, for the right problem we develop a tracking algorithm.

The rest of the paper is organized as follows: In Section 2 we consider the differential problem and we apply the Landau's transformation. In Section 3 we construct a second-order approximation of the problem in the left, right domain and in the interface. In Section 4 we present numerical results.

THE DIFFERENTIAL PROBLEM

We consider the dimensionless solidification problem

$$\frac{\partial v}{\partial t} = k\left(\frac{\partial^2 v}{\partial r^2} + \frac{2}{r}\frac{\partial v}{\partial r}\right), \quad 0 < r < R(t), \tag{1}$$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}, \quad R(t) < r < 1 \tag{2}$$

with fixed boundary conditions

$$\frac{\partial v}{\partial r}|_{r=0} = 0, \quad v(R(t), t) = 0; \tag{3}$$

$$u(R(t), t) = 0, \quad u(1, t) = -1, \tag{4}$$

moving boundary condition (the Stefan condition)

$$\frac{\partial u}{\partial r} - k\frac{\partial v}{\partial r} = \beta\frac{dR}{dt}, \text{ on } r = R(t) \tag{5}$$

and initial conditions

$$v(r,0) = v_0(r); \quad R(0) = 1. \tag{6}$$

Here $\boldsymbol{u(r,t)}$ and $\boldsymbol{v(r,t)}$ are the temperature fields in the the solid and liquid respectively, $\boldsymbol{r}$ is traditional distance, $\boldsymbol{t}$ represents time and $\boldsymbol{r} = \boldsymbol{R(t)}$ describes the location of the solid-melt interface. The three parameters in the problem are the dimensionless initial temperature v_0, the Stefan number β and the ratio of thermal diffusivities k.

The two-phase problem (1)-(6) is highly nonlinear with no known exact analytical solution. We put:

$$\xi = r/R(t), \quad v = v(r,t) = V(\xi, t), \quad u = u(r,t) = U(\xi, t).$$

Then the equations (1)-(2) are converting into

$$\frac{\partial V}{\partial t} = \frac{k}{R^2(t)}\left(\frac{\partial^2 V}{\partial \xi^2} + \frac{2}{\xi}\frac{\partial V}{\partial \xi}\right) + \xi\frac{\dot{R}(t)}{R(t)}\frac{\partial V}{\partial \xi}, \quad 0 < \xi < 1, \tag{7}$$

$$\frac{\partial U}{\partial t} = \frac{1}{R^2(t)}\left(\frac{\partial^2 U}{\partial \xi^2} + \frac{2}{\xi}\frac{\partial U}{\partial \xi}\right) + \xi\frac{\dot{R}(t)}{R(t)}\frac{\partial U}{\partial \xi}, \quad 1 < \xi < \frac{1}{R(t)}; \tag{8}$$

the boundary conditions (3)-(4) into

$$\frac{\partial V}{\partial \xi}|_{\xi=0} = 0, \quad V(1,t) = 0, \quad U(1,t) = 0, \quad U\left(\frac{1}{R(t)}, t\right) = -1. \tag{9}$$

The moving boundary condition (5) takes the form

$$\frac{\partial U}{\partial \xi} - k\frac{\partial V}{\partial \xi} = \beta\dot{R}(t)R(t) \ on \ \xi = 1, \tag{10}$$

and the initial conditions (6)

$$V(\xi, 0) = v_0(\xi R(t)), \quad R(0) = 1. \tag{11}$$

NUMERICAL METHOD

In this section we discretize the transformed equations (7)-(11).

1. Left domain

First, we will construct a second-order approximation of the problem (7)-(11). By Lopital's rule, we have

$$\lim_{\xi\to 0}\frac{\partial V}{\partial \xi} = 0, \quad \lim_{\xi\to 0}\frac{1}{\xi}\frac{\partial V}{\partial \xi} = \lim_{\xi\to 0}\frac{\partial^2 V}{\partial \xi^2} = \frac{\partial^2 V}{\partial \xi^2}|_{\xi=0}.$$

Then, it follows from equation (7)

$$\frac{\partial V}{\partial t}\Big|_{\xi=0} = \frac{3k}{R^2(t)}\frac{\partial^2 V}{\partial \xi^2}\Big|_{\xi=0} . \qquad (12)$$

Therefore,

$$\frac{\partial^2 V}{\partial t \partial \xi} = \frac{k}{R^2(t)}\left(\frac{\partial^3 V}{\partial \xi^3} - \frac{2}{\xi^2}\frac{\partial V}{\partial \xi} + \frac{2}{\xi}\frac{\partial^2 V}{\partial \xi^2}\right) + \frac{\dot{R}(t)}{R(t)}\frac{\partial V}{\partial \xi} + \xi\frac{\dot{R}(t)}{R(t)}\frac{\partial^2 V}{\partial \xi^2}.$$

Next,

$$\frac{\partial^3 V}{\partial \xi^3}\Big|_{\xi=0} = 2\lim_{\xi\to 0}\frac{\frac{1}{\xi}\frac{\partial V}{\partial \xi} - \frac{\partial^2 V}{\partial \xi^2}}{\xi} = 2\lim_{\xi\to 0}\left(-\frac{1}{\xi^2}\frac{\partial V}{\partial \xi} + \frac{1}{\xi}\frac{\partial^2 V}{\partial \xi^2} - \frac{\partial^3 V}{\partial \xi^3}\right).$$

Hence,

$$\frac{\partial^3 V}{\partial \xi^3}\Big|_{\xi=0} = 2\lim_{\xi\to 0}\frac{\frac{\partial^2 V}{\partial \xi^2} - \frac{1}{\xi}\frac{\partial V}{\partial \xi} - \frac{\partial^3 V}{\partial \xi^3}}{\xi},\ wich\ \ implies\ \ \lim_{\xi\to 0}\frac{\partial^3 V}{\partial \xi^3} = 0.$$

Let introduce the mesh $w_h = \{0 < \xi_1 < \ldots < \xi_{N-1} < 1\},\ \xi_0 = 0,\ \xi_N = 1,\ \overline{w}_h = w_h \cup \{\xi_0\} \cup \{\xi_N\}$ and $h_i = \xi_i - \xi_{i-1},\ \hbar_i = 0.5(h_i + h_{i+1})$ [5]. Now, we have

$$\frac{V(h_1,t) - V(-h_1,t)}{2h_1} = \frac{\partial V}{\partial \xi}\Big|_{\xi=0} + \frac{h_1^2}{6}\frac{\partial^3 V}{\partial \xi^3}\Big|_{\xi=0} + O(h_1^4).$$

Using $V(-h_1,t) = V(h_1,t) + O(h_1^5)$ and (12) we get

$$\frac{\partial V}{\partial t}\Big|_{\xi=0} = \frac{6k}{h_1^2 R^2(t)}\big(V(h_1,t) - V(0,t)\big) + O(h_1^3). \qquad (13)$$

From (7) we obtain

$$\frac{\partial V}{\partial t}\Big|_{\xi_i} = \frac{k}{\hbar_i R^2(t)}\left[\left(1 + \frac{h_i}{\xi_i}\right)\frac{V_{i+1} - V_i}{h_{i+1}} - \left(1 - \frac{h_{i+1}}{\xi_i}\right)\frac{V_i - V_{i-1}}{h_i}\right] \qquad (14)$$

$$+ \frac{\xi_i \dot{R}(t)}{2\hbar_i R(t)}\left[h_i\frac{V_{i+1} - V_i}{h_{i+1}} + h_{i+1}\frac{V_i - V_{i-1}}{h_i}\right] + O(\hbar_i^2),\ 1 \le i \le N-1,$$

where $V_i = V(\xi_i, t)$.

We require

$$1/R(t_j) = 1 + jh. \qquad (15)$$

We differentiate $\frac{1}{R(t)}$ in $t = t_j$ and equalize to the angular factor of the segment connected the points $((j-1)h, t_{j-1})$ and (jh, t_j), to obtain

$$-\dot{R}(t)/R^2(t_j) = h/\tau_j. \qquad (16)$$

We discretize (13) as follows:

$$C_0^j V_0^j - B_0^j V_1^j = F_0^j,\ C_0^j = 1 + 6\sigma k\tau_j\left(\frac{1+jh}{h_1}\right)^2,\ B_0^j = 6\sigma k\tau_j\left(\frac{1+jh}{h_1}\right)^2,$$

$$F_0^{\,j} = \left(1 - 6(1-\sigma)k\tau_j\left(\frac{1+(j-1)h}{h_1}\right)^2\right)V_0^{\,j-1} + 6(1-\sigma)k\tau_j\left(\frac{1+(j-1)h}{h_1}\right)^2 V_1^{\,j-1}.$$

Further, we use the Crank-Nicalson scheme to obtain from (14)

$$-A_i^jV_{i-1}^j + C_i^jV_i^j - B_i^jV_{i+1}^j = F_i^j,\ 1 \le i \le N-1.$$

$$A_i^j = \frac{\sigma}{h_i\hbar_i}\left(k\tau_j\left(1-\frac{h_{i+1}}{\xi_i}\right)(1+jh)^2 + \frac{h_{i+1}\xi_i}{2(1/h+j)}\right),$$

$$C_i^j = 1 + \frac{\sigma}{h_ih_{i+1}}\left(2k\tau_j(1+jh)^2\left(1+\frac{h_i-h_{i+1}}{\xi_i}\right) + \frac{\xi_i(h_{i+1}-h_i)}{1/h+j}\right),$$

$$B_i^j = \frac{\sigma}{h_{i+1}\hbar_i}\left(k\tau_j\left(1+\frac{h_i}{\xi_i}\right)(1+jh)^2 - \frac{h_i\xi_i}{2(1/h+j)}\right),$$

$$F_i^j = \frac{(1-\sigma)\tau_j}{h_i\hbar_i}\left(k\left(1-\frac{h_{i+1}}{\xi_i}\right)(1+(j-1)h)^2 + \frac{h_{i+1}\xi_i}{2\tau_{j-1}(1/h+j-1)}\right)V_{i-1}^{j-1}$$

$$+\left[1 - \frac{(1-\sigma)\tau_j}{h_ih_{i+1}}\left(2k(1+(j-1)h)^2\left(1+\frac{h_i-h_{i+1}}{\xi_i}\right) + \frac{\xi_i(h_{i+1}-h_i)}{\tau_{j-1}(1/h+j-1)}\right)\right]V_i^{\,j-1}$$

$$+\frac{(1-\sigma)\tau_j}{h_{i+1}\hbar_i}\left(k\left(1+\frac{h_i}{\xi_i}\right)(1+(j-1)h)^2 - \frac{h_i\xi_i}{2\tau_{j-1}(1/h+j-1)}\right)V_{i+1}^{j-1}.$$

We use the right Thomas's method to find V_{N-1}^j:

$$V_i^j = \alpha_{i+1}^lV_{i+1}^j + \beta_{i+1}^l,\ 1 \le i \le N-1.$$

$$\alpha_{i+1}^l = \frac{B_i^j}{C_i^j - A_i^j\alpha_i^l},\ \alpha_1^l = \frac{B_0^j}{C_0^j}\ 1 \le i \le N-1,$$

$$\beta_{i+1}^l = \frac{F_i^j + \beta_i^lA_i^j}{C_i^j - A_i^j\alpha_i^l},\ \beta_1^l = \frac{F_0^j}{C_0^j},\ 1 \le i \le N-1.$$

$$V_{N-1}^j = \alpha_N^lV_N^j + \beta_N^l = \beta_N^l.$$

2. Right Domain

Let $\xi_i = 1+ih,\ 0 \le i \le j,\ U_0^j = 0,\ U_j^j = -1.$ For $j \ge 1$

$$\frac{\partial U}{\partial t}|_{\xi_i} = \frac{1}{h^2R^2(t)}\left[\frac{\xi_{i+1}}{\xi_i}(U_{i+1}-U_i) - \frac{\xi_{i-1}}{\xi_i}(U_i-U_{i-1})\right] \tag{17}$$

$$+\xi_i\frac{\dot{R}(t)}{R(t)}\frac{U_{i+1}-U_{i-1}}{2h} + O(h^2),\ 1 \le i \le j-1.$$

We discretize (17) as follows:

$$-A_i^j U_{i-1}^j + C_i^j U_i^j - B_i^j U_{i+1}^j = F_i^j,\ 1 \le i \le j-1.$$

$$A_i^j = \sigma\left(\tau_j\left(1-\frac{h}{1+ih}\right)(1/h+j)^2 + \frac{1+ih}{2(1+jh)}\right),$$

$$C_i^j = 1 + 2\sigma\tau_j (1/h+j)^2,$$

$$B_i^j = \sigma\left(\tau_j\left(1+\frac{h}{1+ih}\right)(1/h+j)^2 - \frac{1+ih}{2(1+jh)}\right),$$

$$F_i^j = (1-\sigma)\tau_j\left(\left(1-\frac{h}{1+ih}\right)(1/h+j-1)^2 + \frac{1+ih}{2\tau_{j-1}(1+(j-1)h)}\right)U_{i-1}^{j-1}$$

$$+\left(1-2(1-\sigma)\tau_j(1/h+j-1)^2\right)U_i^{j-1}$$

$$+(1-\sigma)\tau_j\left(\left(1+\frac{h}{1+ih}\right)(1/h+j-1)^2 - \frac{1+ih}{2\tau_{j-1}(1+(j-1)h)}\right)U_{i+1}^{j-1}.$$

We use the left Thomas method to find U_1^j.

$$U_{i+1}^j = \alpha_{i+1}^r U_i^j + \beta_{i+1}^r,\ 1 \le i \le j-1.$$

$$\alpha_i^r = \frac{A_i^j}{C_i^j - B_i^j \alpha_{i+1}^r},\ \alpha_j^r = 0,\ 1 \le i \le j-1,$$

$$\beta_i^r = \frac{F_i^j + \beta_{i+1}^r B_i^j}{C_i^j - B_i^j \alpha_{i+1}^R},\ \beta_j^r = -1,\ 1 \le i \le j-1.$$

$$U_1^j = \alpha_1^r U_0^j + \beta_1^r = \beta_1^r.$$

3. Interface

For $\xi = 1$ we have the next discrete equation:

$$\frac{\partial U}{\partial \xi}\Big|_{1^+} = \left(1 + 1/h + \frac{R(t)\dot{R}(t)}{2}\right)U_1 + O(h^2),$$

$$k\frac{\partial V}{\partial \xi}\Big|_{1^-} = -\left(k(1/h_N - 1) - \frac{R(t)\dot{R}(t)}{2}\right)V_{N-1} + O(h_N^2).$$

From (10) we obtain

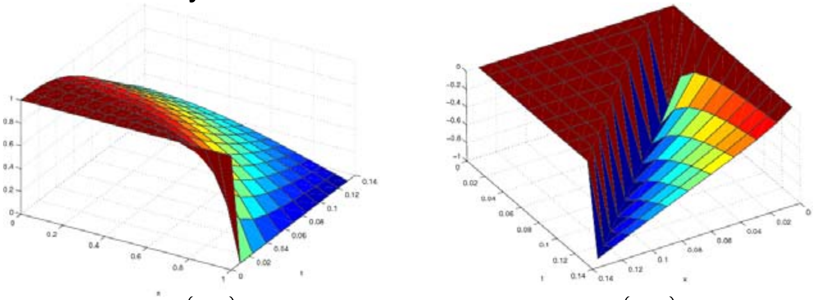
$$R(t)\dot{R}(t) = \frac{2\left[(1+1/h)U_1 + k(1/h_N - 1)V_{N-1}\right]}{2\beta + V_{N-1} - U_1}. \qquad (18)$$

After integration of (18) and use of (15) and (16), we get

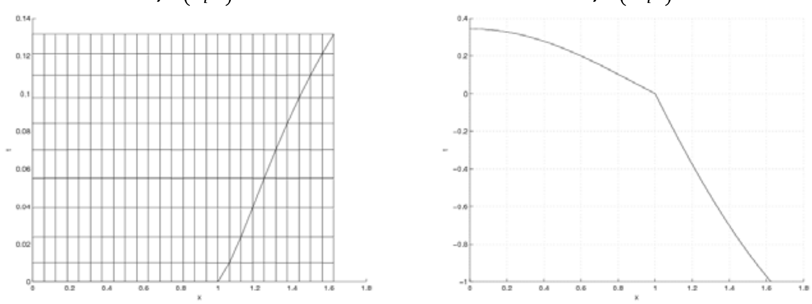
$$\tau_j = \frac{(2\beta + V_{N-1} - U_1)\left(1/(1+jh)^2 - (1/(1+(j-1)h)^2\right)}{4\left[(1+1/h)U_1 + k(1/h_N - 1)V_{N-1}\right]}.$$

NUMERICAL RESULTS

We present the numerical solution of $\{V_i^j\}$, $\{U_i^j\}$, the moving boundary for $\beta = 1$ and for $k = 1$ in Fig. 1 on the uniform mesh and for $k = 2^{-6}$ in Fig. 2 on the Shishkin's mesh. The figures illustrate the efficiency of the Shishkin's mesh.

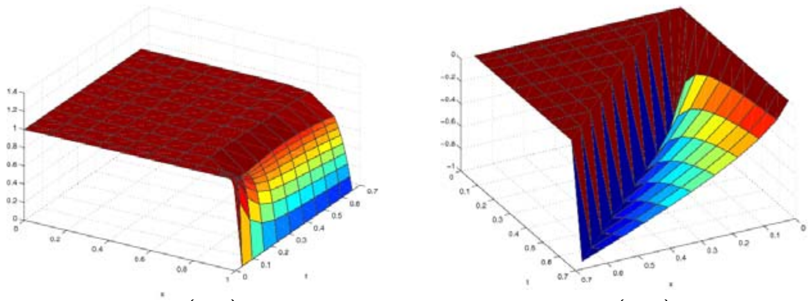


a) $\{V_i^j\}$ b) $\{U_i^j\}$

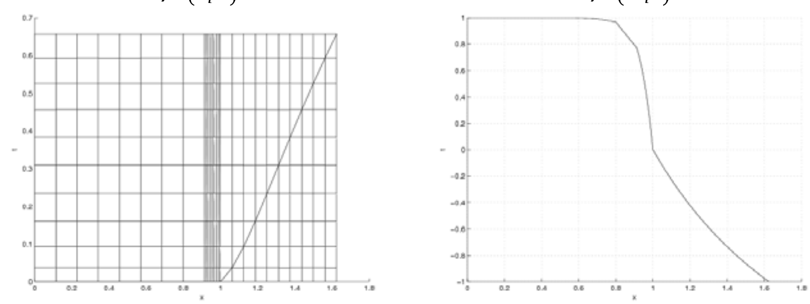


c) Mesh and curve d) V and U on 10-th time layer

Table 1: N=16, j=10, k=1.



a) $\{V_i^j\}$ b) $\{U_i^j\}$



c) Mesh and curve d) V and U on 10-th time layer

Table 2: N=16, j=10, $\boldsymbol{k} = 2^{-6}$.

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ЧИСЛЕНО РЕШАВАНЕ НА ДВУФАЗОВА ЗАДАЧА НА СТЕФАН ЗА СФЕРА

Иванка Ангелова

Русенски университе "Ангел Кънчев"

Резюме: В статията се разглежда двуфазова задача на Стефан за сфера в случай на малка дифузия и голямо число на Стефан. Комбинирани са трансформация на Ландау и метод на Гупта и Кумар с променлива стъпка по времето за численото решаване на задачата. Изведена е схема от втори ред на апроксимация относно пространствената променлива и е предложен алгоритъм за решаване на проблема. Използвана е мрежа на Шишкин в областта на малката дифузия. Обсъдени са числените експерименти.

Ключови думи: Двуфазова задача на Стефан за сфера, задача със свободна граница, диференчна схема, мрежа на Шишкин.

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