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"MATHEMATICS, INFORMATICS AND PHYSICS"

VOLUME 10

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ON THE EXISTENCE OF MULTIPLE PERIODIC SOLUTIONS OF FOURTH-ORDER SEMILINEAR DIFFERENTIAL EQUATIONS

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Abstract: *This paper is focused on periodic solutions of fourth-order nonautonomous semi-linear parabolic equations arising in the dynamics of populations. These equations play also an important role in modeling bi-stable systems, related to studying spatial patterns. Analytical results on the existence of multiple solutions have been presented, using the theorem for minimization and Clark's theorem.*

Keywords: Fourth-order ODE, periodic solutions, minimization theorem, Clark's theorem.

INTRODUCTION

Mathematical models are used to study a number of phenomena, observed in complex natural systems. The so called model equations play a key role in those models. Historically, in 1936-1939 Kolmogorov-Petrovski-Piskunov's equations [8] appear for studying the evolution of species, and Brugers' equations for studying the processes of distribution of different types of flat waves.

Later in 1977-1987, a fourth order model equation was introduced, now known as the extended Fisher - Kolmogorov (EFK) equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \gamma \frac{\partial^4 u}{\partial x^4} + f(u), \quad f(u) = u - u^3 , \gamma > 0 ,$$

proposed by Coullet, Elphick & Repaux [3] and Dee & Saarloos [4]. The EFK equation appeared when different systems were described such as phase transition model in two-dimensional system close to the point of Lifshitz [7], [17]. Stationary solutions of the autonomous equation have been extensively studied in many works of Peletier and his collaborators [10,11,12,13,14]. Existence of stationary periodic solutions and homoclinics of the non-autonomous equation has been obtained by J. Chaparova, L. Sanchez, S. Tersian, T.Gyulov [5,6,16].

This paper focuses on the non-autonomous semi-linear ODE

$$u^{iv} - mu'' + a(x)u|u|^{p-2} - b(x)u|u|^{q-2} = 0 , \tag{1}$$

subject to boundary conditions

$$u(0) = u''(0) = u(L) = u''(L) = 0 . \tag{2}$$

In our previous works [1, 2] the problem (1)-(2) is examined in case $1 < p < q , q > 2$.

We suppose $a(x)$ and $b(x)$ are positive, even, continuous $2L$-periodic functions,

$m > 0$ and $1 < q < p$.

The boundary value problem (1)-(2) has a variational structure and its solutions could be obtained as critical points of the energy functional

$$J(u) = \int_0^L \left[\frac{1}{2} u''^2 + \frac{m}{2} u'^2 + \frac{1}{p} a(x)|u|^p - \frac{1}{q} b(x)|u|^q \right] dx, \tag{3}$$

in the Sobolev space

$$X = H^2(0,L) \cap H_0^1(0,L)$$
$$= \left\{ u \in L^2(0,L) : u' \in L^2(0,L), u'' \in L^2(O,L), u(0) = u(L) = 0 \right\}.$$

The space X is Hilbert space with the scalar product

$$(u,v) = \int_0^L (u''v'' + u'v' + uv) dx, \quad u, v \in X,$$

and norm $\|u\| = \sqrt{(u,u)}$. By the Poincaré inequality

$$\int_0^L u^2 dx \le \frac{L^2}{\pi^2} \int_0^L (u')^2 dx, \quad u \in X,$$

$$\|u\|_1 = \left(\int_0^L \left[(u'')^2 + m(u')^2 \right] dx \right)^{1/2}$$

is an equivalent norm in X. For simplicity we will denote $\|.\|_1$ by $\|.\|$.

EXISTENCE RESULTS

We find the critical points of the functional using the minimization theorem:

Theorem (Minimization theorem [9]). Let J be a functional bounded from below in the reflexive Banach space X. Also let J be weakly lower semi-continuous X and possess a bounded minimizing sequence. Then there is an element $u \in X$, such that $J(u) = \inf_X J$.

Proposition 1. Let $1 < q < p$. Then the functional $J(u)$ is bounded from below on X and there exists a minimizer $u \in X$ of the functional J, i. e. $J(u) = \inf_X J$.

Proof. First we show that the functional is bounded from below. Let $a(x) \geq a_1 > 0$, $b(x) \leq b_2$. Then for every $u \in X$ we have

$$J(u) \geq \frac{1}{2}\|u\|^2 + \frac{a_1}{p}\int_0^L |u|^p\,dx - \frac{b_2}{q}\int_0^L |u|^q\,dx \quad .$$

Denote by $f(u)$ the function $f(u) = \frac{a_1}{p}|u|^p - \frac{b_2}{q}|u|^q$. Then

$$f'(u) = a_1 u|u|^{p-2} - b_2 u|u|^{q-2}$$
$$= u|u|^{q-2}\left(a_1|u|^{p-q} - b_2\right).$$

The minimum of the function $f(u)$ is obtained for $|u_1| = \left(\frac{b_2}{a_1}\right)^{1/(p-q)}$ and it is

$$f_{\min}(u_1) = \left(\frac{b_2^p}{a_1^q}\right)^{1/(p-q)} . \frac{(q-p)}{pq} = k .$$

Taking into account the fact that $1 < q < p$, it follows that $k < 0$.
Thus for any $u \in X$ we obtain

$$J(u) \geq \frac{1}{2}\|u\|^2 + kL \geq kL, \tag{4}$$

hence $J(u)$ is bounded from below.

From the last inequality we obtain $J(u) \to \infty$ as $\|u\| \to \infty$ which means that the functional is coercive, i.e. it possesses a bounded minimizing sequence (u_n).

Next we show that the functional is weakly lower semi-continuous. We present the functional as a sum of two functionals,

$$J(u) = \frac{1}{2}\|u\|^2 + \int_0^L \left[\frac{a(x)}{p}|u|^p - \frac{b(x)}{q}|u|^q\right]dx = \Phi_1(u) + \Phi_2(u).$$

$\Phi_1(u)$ is a convex continuous functional. Therefore, it is weakly lower semi-continuous.

Since (u_n) is bounded we assume that $u_n \rightharpoonup u$ weakly. By Sobolev embedding

theorem it follows that $u_n \to u$ in $C^1[0,L]$. Then $\Phi_2(u_n) \to \Phi_2(u)$, i.e. $\Phi_2(u)$ is weakly continuous.

Thus the functional $J(u)$ is weakly lower semi-continuous in X. The existence of the minimizer $u \in X$ such that $J(u) = \inf_X J$ follows by the general minimization theorem.

Proposition 2. Let $1 < q < p$, $q < 2$ **.** Then the minimizer of the functional J is the non-zero function, which is a non-zero solution of problem (1)-(2).

Proof. We use the test function $u_0(x) = t.\sin\frac{\pi x}{L}$ which depends on a real parameter $t > 0$. We have

$$J(u_{0(x)}) = \frac{t^2\pi^4}{2L^4}\int_0^L \frac{1-\cos\frac{2\pi x}{L}}{2}dx + \frac{mt^2\pi^2}{2L^2}\int_0^L \frac{1+\cos\frac{2\pi x}{L}}{2}dx$$

$$+\frac{t^p}{p}\int_0^L a(x)\left(\sin\frac{\pi x}{L}\right)^p dx - \frac{t^q}{q}\int_0^L b(x)\left(\sin\frac{\pi x}{L}\right)^q dx.$$

Denote

$$c_1 = \frac{\pi^2\left(\pi^2 + 2mL^2\right)}{4L^3} > 0,\ c_2 = \frac{1}{p}\int_0^L a(x)\left(\sin\frac{\pi x}{L}\right)^p dx > 0 \text{ and}$$

$$c_3 = \frac{1}{q}\int_0^L b(x)\left(\sin\frac{\pi x}{L}\right)^q dx > 0.$$

Thus for a sufficiently small $t > 0$ we have

$$J(u_0) = t^q\left(c_1 t^{2-q} + c_2 t^{p-q} - c_3\right) < 0,$$

since $q < 2$ and $1 < q < p$.

As $\inf_X J < J(u_0) < 0 = J(0)$ it follows that the minimizer of J is a nontrivial function, which ends the proof.

We use the following theorem to prove the main result in this paper:

Theorem(Clark [15]). Let X be a real Banach space with a functional $J \in C^1(X, R)$ bounded from below, even and satisfying the Palais-Smale condition. Suppose $J(0) = 0$. Let there be a set $K \subset X$ such that K is homeomorphic to the sphere S^{n-1} in R^n, $n \in N$, by an odd map, and $\sup_K J < 0$. Then the functional J possesses at least n distinct pairs of critical points.

Theorem 1. Let $1 < q < p$ and $q < 2$. Then problem (1)-(2) has infinitely many pairs of solutions.

Proof. For the functional J we know that $J \in C^1(X, R)$, J is even, $J(0) = 0$ and it is bounded from below on X. To satisfy the hypotheses of Clark's theorem, it remains the Palais-Smale condition (PS) and the geometric condition to be proved.

Let (u_n) be a (PS) – sequence of J, i. e.

$(J(u_n))$ be bounded and $J'(u_n) \to 0$ as $n \to \infty$.

We show that the sequence (u_n) is bounded. Indeed from (4), for $M = -kL > 0$, we have

$$J(u) \geq \frac{1}{2}\|u\|^2 - M \Rightarrow J(u_n) + M \geq \frac{1}{2}\|u_n\|^2$$

Since $(J(u_n))$ is bounded there is $C > 0$ such that $|J(u_n)| \leq C$. Then

$$C + M \geq J(u_n) + M \geq \frac{1}{2}\|u_n\|^2.$$

Thus the sequence (u_n) is bounded, and we may assume (going to a subsequence if necessary) that $u_n \rightharpoonup u$ weakly in X.

By Sobolev embedding theorem $u_n \to u$ in $C^1[0, L]$ and $\|u_n\| \to \|u\|$ (see [2]).

Hence $\|u_n\| \to \|u\|$ and $u_n \rightharpoonup u$ in X implies $u_n \to u$ in X.

We show that $\sup_K J(u) < 0$ for $\forall \eta \in N$, where

$$K = \left\{ \lambda_1 \sin\frac{\pi x}{L} + \lambda_2 \sin\frac{2\pi x}{L} + \ldots + \lambda_\eta \sin\frac{\eta\pi x}{L} : \lambda_1^2 + \lambda_2^2 + \ldots + \lambda_\eta^2 = \rho^2 \right\}$$

It is clear that the odd mapping $H : K \to S^{\eta-1}$, defined by

$$H\left(\lambda_1 \sin\frac{\pi x}{L} + \ldots + \lambda_\eta \sin\frac{\eta\pi x}{L}\right) = \left(-\frac{\lambda_1}{\rho}, \ldots, -\frac{\lambda_\eta}{\rho}\right),$$

is a homeomorphism between K and $S^{\eta-1}$, [2].

The set K is a subset of the finite-dimensional space

$$X_\eta = Span\left\{\sin\frac{\pi x}{L}, \sin\frac{2\pi x}{L}, \ldots, \sin\frac{\eta\pi x}{L}\right\},$$

equipped with the norm

$$\left\|\lambda_1 \sin\frac{\pi x}{L} + \lambda_2 \sin\frac{2\pi x}{L} + \ldots + \lambda_n \sin\frac{\eta\pi x}{L}\right\|_{X_\eta}^2 = \lambda_1^2 + \lambda_2^2 + \ldots + \lambda_\eta^2.$$

For each $w \in K$ we have

$$\frac{1}{2}\int_0^L (w'')^2 dx = \frac{L}{4}\left(\frac{\pi}{L}\right)^4 \sum_{k=1}^{\eta} \lambda_k^2 k^4$$

$$\leq \frac{L}{4}\left(\frac{\eta\pi}{L}\right)^4 \sum_{k=1}^{\eta} \lambda_k^2 = \frac{L}{4}\left(\frac{\eta\pi}{L}\right)^4 \|w\|_{X_\eta}^2,$$

$$\frac{m}{2}\int_0^L (w')^2 dx = \frac{mL}{4}\frac{\pi^2}{L^2} \sum_{k=1}^{\eta} \lambda_k^2 k^2$$

$$\leq \frac{mL}{4}\left(\frac{\eta\pi}{L}\right)^2 \sum_{k=1}^{\eta} \lambda_k^2 = \frac{mL}{4}\left(\frac{\eta\pi}{L}\right)^2 \|w\|_{X_\eta}^2,$$

$$\frac{1}{p}\int_0^L a(x)|w|^p dx \leq \frac{a_2}{p}\|w\|_{L^p}^p, \text{ and } \frac{1}{q}\int_0^L b(x)|w|^q dx \geq \frac{b_1}{q}\|w\|_{L^q}^q.$$

Thus

$$J(w) \leq \|w\|_{X_\eta}^q \left(k_1\|w\|_{X_\eta}^{2-q} + k_2\|w\|_{X_\eta}^{p-q} - k_3\right),$$

since from the equivalency of the norms in finite-dimensional spaces for each $r > 1$ there are $k_2(\eta)>0$, $k_3(\eta)>0$, such that

$$k_3(\eta)\|w\|_{X_\eta} \le \|w\|_{L^r} \le k_2(\eta)\|w\|_{X_\eta},\ \forall w \in X_\eta .$$

We choose $\|w\|_{X_\eta}$ sufficiently small and we obtain for $1<q<2$ и $q<p$

$$J(w)<0.$$

The functional J satisfies all hypotheses of Clark's theorem. Hence J has at least η distinct pairs of critical points. Since η is arbitrary, J has infinitely many pairs of critical points which are solutions of (1)-(2).

CONCLUSION

In this work we establish existence of infinitely many solutions of the boundary value problem (1)-(2). Using appropriate extension we obtain infinitely many $2L-$periodic solutions of the equation (1) which are antisymmetric with respect to $x=0$ and $x=L$, namely taking the $2L-$periodic extension of the odd extension

$$\bar{u}(x)=\begin{cases} u(x), & x\in[0,L], \\ -u(-x), & x\in[-L,0), \end{cases}$$

of the solution $u(x)$ for the problem (1)-(2).

The results obtained extend earlier existence results for fourth-order semi-linear ODEs.

REFERENCES

[1] Chaparova J., E.Kalcheva, M.Koleva, Numerical investigation of multiple periodic solutions of fourth-order semi-linear ODE, Amer.Inst. of Physics Conference Proceedings, 1497, 152-159 (2012).

[2] Chaparova J., Kalcheva E., Koleva M. Multiple Periodic Solutions of a Fourth Order Differential Equation via the Symmetric Mountain Pass Theorem. Proc. of the University of Ruse “ A. Kanchev”, vol. 51, book 6.1, Ruse, 2012, pp.28-34.

[3] Coullet P., Elphick C. & Repaux D. Nature of spatial chaos, Phys. Rev. Lett. 58:431-434, (1987).

[4] Dee G.T., & W. van Saarloos, Bistable systems with propagating fronts leading to pattern formation, Phys. Rdv. Lett. 60: 2641-2644, (1988).

[5] Enguica R., A. Gavioli, L. Sanchez, Solutions of second-order and fourth order ODEs on the half-line, NonlinearAnalysis 73(2010), 2968-2927.

[6] Gyulov T., S. Tersian, Existence of trivial and nontrivial solutions for a fourth-order ODE, Electron J. Diff. Eqns.,(2004) no. 41, 1-14.

[7] Hornreich R., Luban M., Shtrikman S. Critical behaviour at the onset of k-space instability on the λ line, Phys. Rev. Lett 35:1678-1681, (1975).

[8] Kolmlgorov A., Petrovski I., Piscounov N. Etude de l'equation de la diffusion avec croissance de la quantite de matiere at son application a unprobleme biologique. Bull. Univ. Moskou, ser. Internat, Sec. A, 1(1937), 1-25.

[9] Mawhin J., Willem M. Critical point theory and Hamiltonian Systems. Springer - Verlag, 1989.

[10] Peletier L., Troy W. Spatial Patterns: Higher order models in physics and mechanics, Birkhlauser, Boston, 2001.

[11] Peletier L., Troy W., Van der Vorst. Stationary soluyions of a fourth-order nonlinear diffusion equation. Differential Equations, 31, 2(1995), 301-314.

[12] Peletier L., Troy W., Spatial patterns described by the extended Fiher-Kolmogorov equation: periodic solutions. SIAM J. Math. Anal., 28, 6 :1317 - 1353, (1997).

[13] Peletier L., Troy W., A topological shooting method and the existence of kinks of the extended Fiher-Kolmogorov equation. Topol. Methods in Nonlinear Anal,6(1995), 331-355.

[14] Peletier L., Troy W., Spatial patterns described by the extended Fiher - Kolmogorov (EFK) equation: Kinks, Diff. Int. Eq., 8 (1995), 1279-1304.

[15] Rabinowitz P. Minimax methods in critical point theory with applications for differential equations, CBMS Regional Conference, 1984.

[16] Tersian S.,J. Chaparova, Periodic and homoclinic solution of extended Fisher-Kolmogorov equations, J. Math. Anal. Appl. 260 (2001), 490-506.

[17] Zimmerman W., Propagating fronts near a Lifshitz point, Phys. Rev. Letters 66:1546.

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ВЪРХУ СЪЩЕСТВУВАНЕ НА БЕЗБРОЙ МНОГО ПЕРИОДИЧНИ РЕШЕНИЯ НА ПОЛУЛИНЕЙНИ ДИФЕРЕНЦИАЛНИ УРАВНЕНИЯ ОТ ЧЕТВЪРТИ РЕД

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Резюме: *В тази статия разглеждаме периодични решения на полулинейни параболични уравнения от четвърти ред. Това са уравнения, описващи динамика на популациите и играещи важна роля при моделирането на биустойчиви системи, свързани с изучаването на пространствени форми. Представени са аналитични резултати за съществуване на безброй много решения, използвайки теоремата за минимизация и теоремата на Кларк.*

Ключови думи: *ОДУ от четвърти ред, периодични решения, теорема за минимизация, теорема на Кларк.*

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