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"MATHEMATICS, INFORMATICS AND PHYSICS"

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BOOK 5

"MATHEMATICS, INFORMATICS AND PHYSICS"

VOLUME 10

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POLYNOMIAL IDENTITIES OF THE 3x3 MATRICES OVER THE FINITE DIMENSIONAL GRASSMANN ALGEBRA

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***Abstract**: In the paper we consider some polynomial identities in the 3 by 3 matrix algebra over the finite dimensional Grassmann algebra, related to the standard polynomial.*
***Keywords**: Matrix algebras, Polynomial identities, Grassmann algebra.*

INTRODUCTION

Let K be a field of characteristic 0 and V be a vector space with ordered basis $\{e_1, e_2, ...\}$.

The associative algebra $G(V)$ with defining relations $e_i e_j + e_j e_i = 0$, for all $i, j = 1,2,...$ is called Grassmann algebra(or Exterior algebra). The basic elements $e_i \in V,\ i = 1,2,...$ are generators of $G(V)$. The elements of the algebra $G(V)$ will be called Grassmann elements. The defining relations allow to rearrange the products of the generators, and $e_{\sigma(i_1)} \dots e_{\sigma(i_n)} = (sign\,\sigma) e_{i_1} \dots e_{i_n}$ for any permutation σ of $i_1, \dots, i_n$.

The basis of $G(V)$ is $B = \{1\} \cup \{e_{i_1} e_{i_2} ... e_{i_m} \mid 1 \le i_1 < i_2 < ... < i_m, m = 1,2,...\}$. The number m defines the length of the basic element $a = e_{i_1} e_{i_2} \dots e_{i_m}$. If a Grassmann element is a linear combination of monomials $e_{i_1} e_{i_2} \dots e_{i_m}$, when m is even, it will be called even Grassmann element. Similarly, when m is odd, it will be called odd Grassmann element.

If dim $V = n$ the corresponding finite dimensional algebra $G(V_n)$ with basis $B_n = \{1, e_1, e_2, e_1 e_2, e_3, e_1 e_3, e_2 e_3, e_1 e_2 e_3, ..., e_1 e_2 ... e_n\}$ will be denoted by G_n.

POLYNOMIAL IDENTITIES

In the paper we consider some polynomial identities in the matrix algebra over the finite dimensional Grassmann algebra.

The polynomial $s_n(x_1, x_2, ..., x_n) = \sum_{\sigma \in S_n} sign(\sigma)\, x_{\sigma(1)} x_{\sigma(2)} ... x_{\sigma(n)}$, where S_n is the symmetric group of degree n is called a standard polynomial. The standard polynomial is multilinear and alternating, namely $s_n(\dots, x, \dots, y, \dots) = -s_n(\dots, y, \dots, x, \dots)$.

In 1950, S. A. Amitsur and J. Levitzki proved [1]: Over any field K, the $n \times n$ matrix algebra $M_n(K)$ satisfies the standard identity of degree $2n$. It does not satisfy polynomial identities of lower degree and, up to a multiplicative constant, $s_{2n} = 0$ is the only mutilinear polynomial identity of degree $2n$ for $M_n(K)$.

The expression $[x_1, x_2] = x_1 x_2 - x_2 x_1$ is called commutator of x_1 and x_2. Inductively a longer commutator is defined by $[x_1, \dots, x_{n-1}, x_n] = [[x_1, \dots, x_{n-1}], x_n], n = 3,4, \dots$.

Proposition 1 [6, Corollary, p.437]. *The T-ideal $T(G)$ is generated by the identity*

$[[x_1, x_2], x_3] = [x_1, x_2, x_3] = 0$.

Proposition 2 [3, Lemma 6.1]. *The algebra* $G(V)$ *satisfies the identity* $s_n^k(x_1, x_2, ..., x_n) = 0$ *for all* $n, k > 1$.

Proposition 3 [5, Exercise 5.3]. *For* $G_n = G(V_n)$ *over* n*-dimensional vector space* V_n, $n > 1$, *all identities follow from the identity* $[x_1, x_2, x_3] = 0$ *and the standard identity* $s_{2p}(x_1, x_2, ..., x_{2p}) = 0$, *where* p *is the minimal integer with* $2p > n$.

PI-degree of an algebra is called the smallest degree of the multilinear identities, which the algebra satisfies.

Proposition 4 [3, Lemma 3.2]. *Let* R *be an algebra with* $PI\deg(R) = r$, *then* $PI\deg(M_n(R)) \geq nr$. *In particular,* $PI\deg(M_n(G)) \geq 3n$.

Proposition 5 [8, Theorem]. *Let* $M_n(G)$ *be the matrix algebra of order* n *over the (infinite dimensional) Grassmann algebra. Then* $M_n(G)$ *has no identities of degree* $4n - 2$.

Proposition 6 [2, Lemma, p.1509]. *The algebra* $M_n(G)$ *satisfies the identity* s_{2n}^k *for some* $k > 1$ *but satisfies neither* s_{2n} *nor identities of the form* s_m^k *for any* k *when* $m < 2n$.

Proposition 7 [4, Proposition 2.1]. *Let* $f_1, \ldots, f_d \in K\langle x_1, \ldots x_m \rangle$ *be elements of the T-ideal of identities of* M_n. *If* $d > \frac{1}{2}n^2 m$, *then* $f_1 \ldots f_d = 0$ *is an identity on* $M_n(G)$.

Some identities to 2 by 2 matrix algebra over the finite dimensional Grassmann algebra, related to the standard polynomial, are investigated in [7].

Proposition 8 [7, Theorem 1]. The algebra $M_2(G_n)$ satisfies the identity $s_4^p(x_1, x_2, x_3, x_4) = 0$, where p is the minimal integer with $2p > n$.

Proposition 9 [5, Exercise 2.8]. *The* $n \times n$ *matrix algebra* $M_n(K)$ *satisfies the identity of algebraicity*

$$d_{n+1}(1, x, x^2, \ldots, x^n, 1, y_1, ..., y_n, 1) = \sum_{\sigma \in Sym\{0,1,\ldots,n\}} sign(\sigma)\, x^{\sigma(0)} y_1 x^{\sigma(1)} y_2 ... x^{\sigma(n-1)} y_n x^{\sigma(n)} = 0,$$

and the identity $s_n([x, y], [x^2, y], \ldots, [x^n, y]) = 0$.

In [9] some identities in $M_3(G)$ are explored.

Proposition 10 [9, Proposition 12]. *The commutator of two symmetric matrices* A_1 *and* A_2 *of type* $\begin{pmatrix} \alpha & \beta & \beta \\ \beta & \alpha & \beta \\ \beta & \beta & \alpha \end{pmatrix}$ *in* $M_3(G)$ *is nilpotent with index of nilpotency* ≤ 3.

The commutator of any two matrices of type $\begin{pmatrix} \alpha & \beta & \beta \\ -\beta & \alpha & \beta \\ -\beta & -\beta & \alpha \end{pmatrix}$ *in* $M_3(G)$ *is nilpotent with index of nilpotency* ≤ 3.

More facts, concerning the index of nilpotency of commutators of length 2 for some types upper triangular matrices over the Grassmann algebra can be seen in [9].

POLYNOMIAL IDENTITIES, RELATED TO THE STANDARD POLYNOMIAL IN $M_3(G_n)$ AND $M_k(G_n)$

Proposition 8 has an analogue in the algebra $M_3(G_n)$.

Theorem 1. The polynomial $s_6^p(x_1, x_2, x_3, x_4, x_5, x_6)$, where p is the minimal integer with $2p > n$, is an identity of $M_3(G_n)$.

Proof. Let $X = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & \alpha_6 \\ \alpha_7 & \alpha_8 & \alpha_9 \end{pmatrix}$ be a matrix with entries Grassmann elements from $G_n = G(V_n)$ and

$$\alpha_i = \alpha_{i_0} + \alpha_{i_1} e_1 + \alpha_{i_2} e_2 + \alpha_{i_3} e_3 + \cdots + \alpha_{i_n} e_n + \alpha_{i_{n+1}} e_1 e_2 + \cdots + \alpha_{i_{2^n-1}} e_1 e_2 \ldots e_n,$$

$i = 1, \ldots, 9,\ a_{i_s} \in K,\ s = 0, 1, \ldots, 2^n - 1.$

We can express X as follows

$$X = \begin{pmatrix} \alpha_{1_0} & \alpha_{2_0} & \alpha_{3_0} \\ \alpha_{4_0} & \alpha_{5_0} & \alpha_{6_0} \\ \alpha_{7_0} & \alpha_{8_0} & \alpha_{9_0} \end{pmatrix} + \sum_{j=1}^{n} \begin{pmatrix} \alpha_{1_j} & \alpha_{2_j} & \alpha_{3_j} \\ \alpha_{4_j} & \alpha_{5_j} & \alpha_{6_j} \\ \alpha_{7_j} & \alpha_{8_j} & \alpha_{9_j} \end{pmatrix} e_j + \cdots +$$

$$+ \begin{pmatrix} \alpha_{1_{n+1}} & \alpha_{2_{n+1}} & \alpha_{3_{n+1}} \\ \alpha_{4_{n+1}} & \alpha_{5_{n+1}} & \alpha_{6_{n+1}} \\ \alpha_{7_{n+1}} & \alpha_{8_{n+1}} & \alpha_{9_{n+1}} \end{pmatrix} e_1 e_2 + \cdots + \begin{pmatrix} \alpha_{1_{2^n-1}} & \alpha_{2_{2^n-1}} & \alpha_{3_{2^n-1}} \\ \alpha_{4_{2^n-1}} & \alpha_{5_{2^n-1}} & \alpha_{6_{2^n-1}} \\ \alpha_{7_{2^n-1}} & \alpha_{8_{2^n-1}} & \alpha_{9_{2^n-1}} \end{pmatrix} e_1 e_2 \ldots e_n.$$

Let $X_i = \begin{pmatrix} \alpha_{1_i} & \alpha_{2_i} & \alpha_{3_i} \\ \alpha_{4_i} & \alpha_{5_i} & \alpha_{6_i} \\ \alpha_{7_i} & \alpha_{8_i} & \alpha_{9_i} \end{pmatrix}$, $i = 0, 1, \ldots, 2^n - 1$. Then

$$X = X_0 + X_1 e_1 + X_2 e_2 + X_3 e_3 + \cdots + X_n e_n + X_{n+1} e_1 e_2 + \cdots + X_{2^n-1} e_1 e_2 \ldots e_n. \qquad (1)$$

We consider six matrices $X^{(k)}$, $k = 1, \ldots, 6$ of type (1). $X^{(k)} = X_0^{(k)} + X_1^{(k)} e_1 + X_2^{(k)} e_2 + X_3^{(k)} e_3 + \cdots + X_n^{(k)} e_n + X_{n+1}^{(k)} e_1 e_2 + \cdots + X_{2^n-1}^{(k)} e_1 e_2 \ldots e_n,$ $k = 1, \ldots, 6$, and transform the standard polynomial, using its multilinearity, as follows:

$$s_6\left(X^{(1)}, \ldots, X^{(6)}\right) =$$

$$s_6(X_0^{(1)} + X_1^{(1)} e_1 + \cdots + X_n^{(1)} e_n + X_{n+1}^{(1)} e_1 e_2 + \cdots + X_{2^n-1}^{(1)} e_1 e_2 \ldots e_n, \ldots,$$

$$X_0^{(6)} + X_1^{(6)} e_1 + \cdots + X_n^{(6)} e_n + X_{n+1}^{(6)} e_1 e_2 + \cdots + X_{2^n-1}^{(6)} e_1 e_2 \ldots e_n) =$$

$$= s_6\left(X_0^{(1)}, X_0^{(2)}, X_0^{(3)}, X_0^{(4)}, X_0^{(5)}, X_0^{(6)}\right) +$$
$$+ \sum_{j=1}^{n}\left[s_6\left(X_j^{(1)}e_j, X_0^{(2)}, X_0^{(3)}, X_0^{(4)}, X_0^{(5)}, X_0^{(6)}\right) + s_6\left(X_0^{(1)}, X_j^{(2)}e_j, X_0^{(3)}, X_0^{(4)}, X_0^{(5)}, X_0^{(6)}\right) +\right.$$
$$+ s_6\left(X_0^{(1)}, X_0^{(2)}, X_j^{(3)}e_j, X_0^{(4)}, X_0^{(5)}, X_0^{(6)}\right) + s_6\left(X_0^{(1)}, X_0^{(2)}, X_0^{(3)}, X_j^{(4)}e_j, X_0^{(5)}, X_0^{(6)}\right) +$$
$$\left.+ s_6\left(X_0^{(1)}, X_0^{(2)}, X_0^{(3)}, X_0^{(4)}, X_j^{(5)}e_j, X_0^{(6)}\right) + s_6\left(X_0^{(1)}, X_0^{(2)}, X_0^{(3)}\ X_0^{(4)}, X_0^{(5)}, X_j^{(6)}e_j\right)\right] + S =$$

$$= s_6\left(X_0^{(1)}, X_0^{(2)}, X_0^{(3)}, X_0^{(4)}, X_0^{(5)}, X_0^{(6)}\right) +$$
$$\sum_{j=1}^{n}\left[s_6\left(X_j^{(1)}, X_0^{(2)}, X_0^{(3)}, X_0^{(4)}, X_0^{(5)}, X_0^{(6)}\right)e_j + s_6\left(X_0^{(1)}, X_j^{(2)}, X_0^{(3)}, X_0^{(4)}, X_0^{(5)}, X_0^{(6)}\right)e_j +\right.$$
$$+ s_6\left(X_0^{(1)}, X_0^{(2)}, X_j^{(3)}, X_0^{(4)}, X_0^{(5)}, X_0^{(6)}\right)e_j + s_6\left(X_0^{(1)}, X_0^{(2)}, X_0^{(3)}, X_j^{(4)}, X_0^{(5)}, X_0^{(6)}\right)e_j +$$
$$\left.+ s_6\left(X_0^{(1)}, X_0^{(2)}, X_0^{(3)}, X_0^{(4)}, X_j^{(5)}, X_0^{(6)}\right)e_j + s_6\left(X_0^{(1)}, X_0^{(2)}, X_0^{(3)}, X_0^{(4)}, X_0^{(5)}, X_j^{(6)}\right)e_j\right] + S.$$

Here S is the sum of products of matrices with entries from K and the basic elements $e_1 e_2 ... e_m,\ m \geq 2$.

Each of the matrices $X_j^{(k)}$ for $k = 1, \ldots, 6$ and $j = 0,1, \ldots, n$ is a matrix with entries from K . Then due to Amitsur-Levitzki theorem it follows

$$s_6\left(X_0^{(1)}, X_0^{(2)}, X_0^{(3)}, X_0^{(4)}, X_0^{(5)}, X_0^{(6)}\right) = 0;$$
$$s_6\left(X_0^{(1)}, \ldots, X_j^{(k)}, \ldots, X_0^{(6)}\right) = 0,\ \forall k = 1, \ldots, 6; j = 0,1, \ldots, n.$$

Then $s_6\left(X^{(1)}, \ldots, X^{(6)}\right) = S$ and $s_6^p = S^p$. Since S is a sum of matrices multiplied by the basic elements $e_1 e_2 ... e_m,\ m \geq 2$, then S^p will be a sum of matrices multiplied by the elements $e_1 e_2 ... e_q,\ q \geq 2p$. If $2p > n$ then in the element $e_1 e_2 ... e_q,\ q \geq 2p > n$ there is at least one repeated generator and $e_1 e_2 ... e_q = 0$. Hence $s_6^p = 0.$ This completes the proof.

We can generalize the above theorem for the matrix algebra $M_k(G_n)$. The additional difficulties are only of technical nature.

Theorem 2. The polynomial $s_{2k}^p\left(x_1, x_2, x_3, x_4, x_5, x_6\right)$, where p is the minimal integer with $2p > n$, is an identity of $M_k(G_n)$.

REFERENCES

[1] Amitsur, S.A., J. Levitzki, *Minimal identities for algebras*, Proc. Amer. Math. Soc. 1 (1950), 449-463.

[2] Berelle, B.A., *Classification theorems for verbally semiprime algebras*, Commun. Algebra, 21(5) (1993), 1505-1512.

[3] Berelle, B.A., A. Regev, *Exponential growth for codimensions of some P.I. algebras*, J. Algebra, 241(1) (2001), 118-145.

[4] Domokos, M., A. Popov, *On the degree of nilpotency of the radical of relatively free algebras*, Math. Pannonica 8/1 (1997), 11-16.

[5] Drensky, V., *Free Algebras and PI Algebras*, Springer, Singapore,1999.

[6] Krakowski, D., A. Regev, The *polynomial identities of the Grassmann algebra*, Trans. Amer. Math. Sos., 181(1973), 429-438.

[7] Mihova, A., Polynomial Identities of the 2x2 Matrices over the Finite Dimensional Grassmann Algebra, Proceedings of the Union of Scientists – Ruse, b.5 Mathematics, Informatics and Physics, vol. 8 (2011), 13-18.

[8] Popov, P., On the minimal degree of the identities of the matrices over the Grassmann algebra, Preprint, Amer.Univ.of Bulgaria, Blagoevgrad, Bulgaria (1997).

[9] Rashkova, Ts., On the nilpotency in matrix algebras with Grassmann entries, Serdica Math. J., vol. 38 (2012), 79-90.

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ПОЛИНОМНИ ТЪЖДЕСТВА В МАТРИЧНАТА АЛГЕБРА ОТ ТРЕТИ РЕД НАД КРАЙНОМЕРНА ГРАСМАНОВА АЛГЕБРА

Антоанета Михова

Русенски университет „Ангел Кънчев”

Резюме*: В статията са разгледани тъждества, свързани със стандартния полином, в матричната алгебра от трети ред над крайномерна Грасманова алгебра.*

Ключови думи*: Матрична алгебра, Полиномни тъждества, Грасманова алгебра.*

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