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BOOK 5

"MATHEMATICS, INFORMATICS AND PHYSICS"

VOLUME 9

CONTENTS

A COMPARISON OF TWO METHODS FOR CALCULATION WITH GRASSMANN NUMBERS

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Abstract: *The paper considers two methods for calculations with Grassmann numbers, their possibilities are described and a comparison between them is made.*
Keywords: *Grassmann algebra, Wedge product, Calculations with Grassmann numbers.*

INTRODUCTION

Let K be a field of characteristic 0 and V be a vector space with ordered basis $\{e_1, e_2, ...\}$.

For initial concepts and statements we refer to [1], [2] and [6].

A vector space R is called an algebra if R is equipped with a binary operation "$*$", called a multiplication such that for any $a,b,c \in R$ and $\forall \alpha \in K$ $(a+b)*c = a*c + b*c$, $c*(a+b) = c*a + c*b$, $\alpha(a*b) = (\alpha a)*b = a*(\alpha b)$.

The Exterior algebra $G(V)$ of V is the associative algebra, generated by the basis of V with defining relations $e_i * e_j + e_j * e_i = 0$, for all $i, j = 1,2,...$. The elements $e_i \in V$, $i = 1,2,...$ are called generators of $G(V)$. The multiplication of Exterior algebra is known as *exterior product* or *wedge product* and usually is denoted by "$\wedge$". The exterior product is the natural fundamental product operation for elements of a vector space. This operation is not closed, because the product of two elements of a vector space is not an element of the same linear space, but the products of its generators form a new linear space which is closed and it is an algebra itself.

The Exterior algebra was first introduced by Herman Grassmann in1844 and after him it is named Grassmann algebra. The basis of $G(V)$ is the set $B = \{1\} \cup \{e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_m} \mid 1 \le i_1 < i_2 < ... < i_m, m = 1,2,...\}$.

Well known properties of exterior product are:

- $a \wedge b = -b \wedge a$, for all $a,b \in V$ - *the exterior product is antisymmetric;*

It follows immediately that

- $a \wedge a = 0$;
- $a \wedge (\lambda b) = \lambda(a \wedge b)$, $a,b \in V$, $\lambda \in K$;
- $(\lambda a) \wedge b = \lambda(a \wedge b)$, $a,b \in V$, $\lambda \in K$;
- $(a+b) \wedge x = a \wedge x + b \wedge x$, $a,b,x \in V$.

The exterior product is useful in many ways. One powerful property of exterior product is its close relation to linear independence of set of vectors: A set $\{v_1, \dots, v_k\}$ of vectors from V is linear independent if and only if $v_1 \wedge v_2 \wedge \dots \wedge v_k \neq 0$.

The wedge product can be used to calculate determinants and volumes of parallelepipeds. For example, if $\det M = \det(v_1, \dots, v_n)$ where $v_i, i = 1,2,\dots,n$ are the columns of M, then $v_1 \wedge \dots \wedge v_n = \det(v_1, \dots, v_n) e_1 \wedge \dots \wedge e_n$, $\det(v_1, \dots, v_n)$ is the volume of the parallelepiped spanned on $\{v_1, \dots, v_n\}$.

Let V_n be a finite dimensional vector space with dimension n and $G_n = G(V_n)$ be the Grassmann algebra over V_n spanned on $B_n = \{1, e_1, e_2, e_1 \wedge e_2, e_3, ..., e_1 \wedge e_2 \wedge ... \wedge e_n\}$.

If a is a basic element and $1 \neq a = e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_m}$, then m is called a length of a.

The elements of Grassmann algebra are called *Grassmann numbers*. If a Grassmann number is represented as a sum only of monomials with even length it is called *even Grassmann number*, and if a Grassmann number is represented as a sum only of monomials with odd length it is called *odd Grassmann number.*

The Grassmann algebra has some important applications in linear geometry, differential geometry, physical theories of fermions and supersymmetry, mechanics. The increasing importance of Grassmann algebra causes a need of computer programs and systems for calculation with Grassmann numbers.

The computational software program *Mathematica*, developed by *Wolfram Research* [7], has added to its packages a new one - *GrasmmanAlgebra* package. This package is written by John Browne (Associative professor at Swinburne University of Technology in Australia) and his students, using the *Mathematica's* programming language. While working on the *GrasmmanAlgebra* package, Browne has also been writing a book "Grassmann Algebra: Exploring applications of extended vector algebra with *Mathematica*"[1]. In 2001 Browne published an incomplete draft of this book in his university home page. In 2009 in connection with the 200th anniversary of Grassmann's birth he published the book on http://sites.google.com/site/Grassmannalgebra/home - the companion site for the book and for the Mathematica-based software package *GrassmannAlgebra.*

In the meantime Ts. Rashkova and A. Mihova using a correspondence, described in [5] between the integers from 0 to $2^n - 1$ and the basic elements of the Grassmann algebra over *n*-dimensional vector space, made a program in *Mathematica* for multiplication of Grassmann numbers. Later the program was used to make a new program for multiplication 2×2 matrices with entries from a finite dimensional Grassmann algebra [4].

This paper considers the mentioned above two ways for working with elements of Grassmann algebra. To use the *Mathematica's* potential is the common between these two programs.

GRASSMANNALGEBRA PACKAGE – A BRIEF DESCRIPTION OF ITS POSSIBILITIES

What the *GrasmmanAlgebra* package do is very well described in [3]. Namely, with the *GrasmmanAlgebra* package we can:

- Set up your own space of any dimension and metric. The default is a three-dimensional Euclidean space.
- Work basis-free or with a basis as appropriate.
- Declare your own scalar symbols: symbols or symbol patterns you want specially interpreted as scalars.
- Declare your own vector symbols: symbols or symbol patterns you want specially interpreted as vectors.
- Apply different Grassmann operations, which could be specified as the complement operation or the exterior, regressive, interior, generalized Grassmann, hypercomplex and Clifford products.

- Manipulate Grassmann expressions and numbers. A Grassmann expression is either a scalar, a Grassmann variable, or the result of a sequence of Grassmann operations or sums of Grassmann expressions.
- Simplify Grassmann expressions using multi-purpose simplification functions.
- Compute functions of several Grassmann numbers.
- Perform Grassmann matrix algebra with the elements being Grassmann expressions and the underlying (element by element) product operation being any of the Grassmann product operations.
- Compute functions of a matrix of Grassmann numbers with distinct eigenvalues.
- Manipulate lists and matrices of Grassmann expressions (where applicable) as easily as single expressions.

As we can see the *GrassmannAlgebra* package can do so many manipulations, and of course it takes long time to study them completely. We are interested mainly in multiplying Grassmann numbers.

Let's see how we can multiply two Grassmann numbers using the package. First of all we have to load it. This can be done by clicking on the *GrassmannAlgebra Palette* bottom or with the function *Needs["GrassmannAlgebra`"*]. The fields below the bottom show the currently declared basis, scalar symbols, and vector symbols.

Since the default basis is spanned on $\{e_1, e_2, e_3\}$ it is easy to introduce the Grassmann numbers $x=2+3e_1+4e_2+5e_1\wedge e_2$ and $y=1+3e_1+4e_2+7e_1\wedge e_2$. We use the function *GrassmannExpandAndSimplify* to find and simplify the exterior product of these two numbers.

In[1]= $x=2+3e_1+4e_2+5e_1\wedge e_2$
Out[1]= $x=2+3e_1+4e_2+5e_1\wedge e_2$
In[2]=$1+3e_1+4e_2+7e_1\wedge e_2$
Out[2]=$1+3e_1+4e_2+7e_1\wedge e_2$
In[3]=GrassmannExpandAndSimplify[$(2+3\ e_1+4\ e_2+5\ e_1\wedge e_2)\ \wedge\ (1+3\ e_1+4\ e_2+7\ e_1\wedge e_2)$]

The product is

Out[3]=$2+9\ e_1+12\ e_2+19\ e_1\wedge e_2$ (1)

If we want to multiply elements from G_4 we have to declare a new basis, and then introduce numbers and use some appropriate functions:

In[4]=DeclareBasis[4]
Out[4]=$\{e_1,e_2,e_3,e_4\}$
In[5]=$x=a+(a+1)\ e_1+a\ e_2+3\ e_1\wedge e_2+(a-1)\ e_3+2\ a\ e_4+a\ e_2\wedge e_4+(2\ a-1)\ e_1\wedge e_2\wedge e_3$
Out[5]=$a+(1+a)\ e_1+a\ e_2+(-1+a)\ e_3+2\ a\ e_4+3\ e_1\wedge e_2+a\ e_2\wedge e_4+(-1+2\ a)\ e_1\wedge e_2\wedge e_3$
In[6]= $y=a+(2\ a-1)\ e_1+2\ e_2+3\ e_1\wedge e_3+2\ a\ e_1\wedge e_3\wedge e_4+(a+2)\ e_4+(-a+1)\ e_1\wedge e_2\wedge e_3$
Out[6]=$a+(-1+2\ a)\ e_1+2\ e_2+(2+a)\ e_4+3\ e_1\wedge e_3+(1-a)\ e_1\wedge e_2\wedge e_3+2\ a\ e_1\wedge e_3\wedge e_4$
In[7]= GrassmannExpandAndSimplify[$x\wedge y$]
Out[7]=$a^2+(a\ (1+a)+a\ (-1+2\ a))\ e_1+(2\ a+a^2)\ e_2+(-1+a)\ a\ e_3+(2\ a^2+a\ (2+a))\ e_4+(4\ a-2\ a^2+2\ (1+a))\ e_1\wedge e_2+(-1+6\ a-2\ a^2)\ e_1\wedge e_3+((1+a)\ (2+a)-2\ a\ (-1+2\ a))\ e_1\wedge e_4+(2-2\ a)\ e_2\wedge e_3+(-4\ a+a^2+a\ (2+a))\ e_2\wedge e_4+(-2+a+a^2)\ e_3\wedge e_4+(-3\ a+(1-a)\ a+a\ (-1+2\ a))\ e_1\wedge e_2\wedge e_3+(3\ (2+a)+a\ (-1+2\ a))\ e_1\wedge e_2\wedge e_4+(6\ a+2\ a^2)\ e_1\wedge e_3\wedge e_4+(-3\ a+2\ (-1+a)\ a-2\ a^2+(2+a)\ (-1+2\ a))\ e_1\wedge e_2\wedge e_3\wedge e_4$

The result needs to be simplified, and we use the next function:

In[8]=GrassmannSimplify[%](% means the last obtained expression)
Out[8]=$a^2+3\ a^2\ e_1+a\ (2+a)\ e_2+(-1+a)\ a\ e_3+a\ (2+3\ a)\ e_4+(2+6\ a-2\ a^2)\ e_1\wedge e_2+(-1+6\ a-2\ a^2)\ e_1\wedge e_3+(2+5\ a-3\ a^2)\ e_1\wedge e_4+(2-2\ a)\ e_2\wedge e_3+2\ (-1+a)\ a\ e_2\wedge e_4+(-2+a+a^2)\ e_3\wedge e_4+(-3+a)\ a\ e_1\wedge e_2\wedge e_3+2\ (3+a+a^2)\ e_1\wedge e_2\wedge e_4+2\ a\ (3+a)\ e_1\wedge e_3\wedge e_4+2\ (-1-a+a^2)\ e_1\wedge e_2\wedge e_3\wedge e_4$ (2)

Let's verify the identity $[x \wedge y - y \wedge x]$ for arbitrary even Grassmann numbers from G_4.

In[9]=DeclareBasis[4]
Out[9]={e_1,e_2,e_3,e_4}
In[10]=DeclareExtraScalarSymbols[a_,b_]
In[11]=x=$a_1+a_4e_1\wedge e_2+a_6e_1\wedge e_3+a_7e_2\wedge e_3+a_{10}e_1\wedge e_4+a_{11}e_2\wedge e_4+a_{13}e_3\wedge e_4+a_{16}e_1\wedge e_2\wedge e_3\wedge e_4$
Out[11]=$a_1+a_4e_1\wedge e_2+a_6e_1\wedge e_3+a_{10}e_1\wedge e_4+a_7e_2\wedge e_3+a_{11}e_2\wedge e_4+a_{13}e_3\wedge e_4+a_{16}e_1\wedge e_2\wedge e_3\wedge e_4$
In[12]=y=$b_1+b_4e_1\wedge e_2+b_6e_1\wedge e_3+b_7e_2\wedge e_3+b_{10}e_1\wedge e_4+b_{11}e_2\wedge e_4+b_{13}e_3\wedge e_4+b_{16}e_1\wedge e_2\wedge e_3\wedge e_4$
Out[12]=$b_1+b_4e_1\wedge e_2+b_6e_1\wedge e_3+b_{10}e_1\wedge e_4+b_7e_2\wedge e_3+b_{11}e_2\wedge e_4+b_{13}e_3\wedge e_4+b_{16}e_1\wedge e_2\wedge e_3\wedge e_4$
In[13]=GrassmannExpandAndSimplify[x∧y-y∧x]
Out[13]=0 (3)

Clearly the GrassmannAlgebra package can do manipulations with numbers with specific coefficients and variable coefficients.

There are many functions in the book related to Grassmann numbers, but in the GrassmannAlgebra package they are not available yet. For example such functions are: CreateGrassmannNumber - creates a Grassmann number with scalar coefficients, CreateMatrixForm - creates a matrix with elements grasmann numbers, MatrixProduct - multiplies matrices, etc.

A PROGRAM USING A CORRESPONDENCE BETWEEN THE INTEGERS AND THE BASIC ELEMENTS OF THE GRASSMANN ALGEBRA (SECOND METHOD)

Let $G_n = G(V_n)$ be the Grassmann algebra over the finite dimensional vector space V_n. Let i be an integer for $0 \le i \le 2^n - 1$ and $i = \alpha_1\,\alpha_2 \ldots \alpha_{n(2)}$, $\alpha_i \in \{0,1\}, i = 1, 2, \ldots, n$ be its binary representation. Each i corresponds to the basic element $e_1^{\alpha_n} \wedge e_2^{\alpha_{n-1}} \wedge \ldots \wedge e_n^{\alpha_1}$, and then any Grassmann number $x \in G_n = G(V_n)$ can be expressed in the form:

$$x = a_0 + a_1e_1 + a_2e_2 + a_3e_1e_2 + a_4e_3 + a_5e_1e_3 + a_6e_2e_3 + a_7e_1e_2e_3 + \cdots + a_{2^n-1}e_1e_2\ldots e_n,$$

$a_i \in K, i = 0,1,\ldots, 2^n - 1.$

This correspondence gives the possibility to work only with the indices of coefficients. Using it a program is done, reported in [5], which multiplies Grassmann numbers,.

To multiply two Grassmann elements we input them as vectors with coordinates the coefficients of the numbers and obtain the product in the same form.

Let's now multiply the same Grassmann numbers of the examples above.

We input x=2+3e_1+4e_2+5$e_1\wedge e_2$ and y=1+3e_1+4e_2+7$e_1\wedge e_2$ in the form

In[1]=x={2,3,4,5}
Out[1]={2,3,4,5}
In[2]=y={1,3,4,7}
Out[2]={1,3,4,7}
In[3]=x∧y
Out[3]={2,9,12,19}. This result means **x**∧**y** =2+9e_1+12e_2+19$e_1\wedge e_2$. (4)

Let x=a+(a+1) e_1+a e_2+3 $e_1\wedge e_2$+(a-1) e_3+2 a e_4+a $e_2\wedge e_4$+(2 a-1) $e_1\wedge e_2\wedge e_3$, and y=a+(2 a-1) e_1+2 e_2+3 $e_1\wedge e_3$+2 a $e_1\wedge e_3\wedge e_4$+(a+2) e_4+(-a+1) $e_1\wedge e_2\wedge e_3$.

In[4]=x={a,a+1,a,3,a-1,0,0,2a-1,2a,0,a,0,0,0,0,0}
Out[4]={a,1+a,a,3,-1+a,0,0,-1+2 a,2 a,0,a,0,0,0,0,0}
In[5]=y={a,2a-1,2,0,0,3,0,1-a,a+2, 0,0,0,0,2a,0,0}
Out[5]= {a,-1+2 a,2,0,0,3,0,1-a,2+a,0,0,0,0,2 a,0,0}
In[6]=x∧y={a^2,a(1+a)+a(-1+2a),2a+a^2,3a+2(1+a)-a(-1+2a),(-1+a)a,3a-(-1+a)(-1+2a),

-2(-1+a),-3a+(1-a)a+a(-1+2 a),2 a^2+a (2+a),(1+a) (2+a)-2 a (-1+2 a),-4 a+a^2+a (2+a),3 (2+a)+a (-1+2 a),(-1+a) (2+a),6 a+2 a^2,0,-3 a-2 (1-a) a-2 a^2+(2+a) (-1+2 a)}

In[6]= Simplify[%]

Out[6]= {a^2,3 a^2,a (2+a),2+6 a-2 a^2,(-1+a) a,-1+6 a-2 a^2,2-2 a,(-3+a) a,a (2+3 a),2+5 a-3 a^2,2 (-1+a) a,2 (3+a+a^2),-2+a+a^2,2 a (3+a),0,2 (-1-a+a^2)}

This means $x \wedge y = a^2+3a^2 e_1 + a(2+a)e_2 + (2+6a-2a^2) e_1 \wedge e_2 + (-1+a)ae_3 + (-1+6a-2a^2)e_1 \wedge e_3 + (2-2a)e_2 \wedge e_3 + (-3+a)a\, e_1 \wedge e_2 \wedge e_3 + a(2+3a)e_4 + (2+5a-3a^2)e_1 \wedge e_4 + 2(-1+a)ae_2 \wedge e_4 + 2(3+a+a^2)\, e_1 \wedge e_2 \wedge e_4 + (-2+a+a^2)\, e_3 \wedge e_4 + 2a(3+a)\, e_1 \wedge e_3 \wedge e_4 + 0\, e_2 \wedge e_3 \wedge e_4 + 2(-1-a+a^2)e_1 \wedge e_2 \wedge e_3 \wedge e_4.$ (5)

It is easy to see that results (1) and (4) are equal, (2) and (5) are equal too.

An advantage of the correspondence is the possibility to work with odd and even grassman numbers. The next functions are very suitable:

DigitCount[*n*, *b*, *d*] - gives the number of *d* digits in the base *b*-representation of *n*.

Random[type, {x_{min}, x_{max}}] – gives a pseudorandom number of specified type, lying in the specified range {x_{min}, x_{max}}. Possible types are: Integer, Real and Complex.

IntegerDigits[*n*, *b*] – gives the digits of *n* in base *b*.

For example we input an odd Grassmann number from $G_6 = G(V_6)$ with coefficients random integer numbers from 0 to 100 as follows:

In[7]=For[i=1,i≤2^6,i++,
If[Mod[DigitCount[i-1,2,1],2]==0,a1[i]=0,a1[i]=Random[Integer,{0,100}]]]

In[8]=x1=Array[a1,2^n]

Out[8]={0,10,20,0,82,0,0,89,28,0,0,16,0,2,22,0,37,0,0,16,0,59,80,0,0,48,69,0,1,0,0,21,23, 0,0,26,0,3,78,0,0,49,16,0,92,0,0,50,0,12,91,0,42,0,0,44,64,0,0,14,0,53,66,0} (6)

Let's again verify the identity $[x \wedge y - y \wedge x]$ for arbitrary even Grassmann numbers from G_4.

In[11]= For[i=1,i≤16,i++,
If[Mod[DigitCount[i-1,2,1],2] ≠0,a[i]=0,]]

In[12]= x=Array[a,2^4]

Out[12]= {a[1],0,0,a[4],0,a[6],a[7],0,0,a[10],a[11],0,a[13],0,0,a[16]}

In[13]= For[i=1,i≤16,i++,
If[Mod[DigitCount[i-1,2,1],2] ≠0,b[i]=0,]]

In[14]= y=Array[b,2^4]

Out[14]= {b[1],0,0,b[4],0,b[6],b[7],0,0,b[10],b[11],0,b[13],0,0,b[16]}

In[15]= $x \wedge y - y \wedge x$

Out[15]= {0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0} (7)

To get the result (7) using the program written took twice less time than to get the result (3) using the *GrassmannAlgebra* package.

This method used in the program allows to work very easy with Grassmann numbers from $G_n = G(V_n)$ when *n* is a large number.

Since we can see only the coefficients and we can't see the basic elements corresponding to them we use the function *IntegerDigits[*n*, 2]* to see the basic element related to a coefficient. For example let's find the basic element corresponding to $\mathbf{a_{123}}$.

In[16]= IntegerDigits[123*,* 2]

Out[16]= {1,1,1,1,0,1,1}. Then the basic element is $e_1e_2e_4e_5e_6e_7$.

As we said in the introduction, the described binary correspondence is used to make a program for multiplying 2x2 matrices over finite dimensional Grassmann algebra.

CONCLUSIONS

The both considered methods for calculations with Grassmann numbers are suitable for working in the field of the Grassmann Algebra:

- to declare the basis is equally easy in the both cases;
- if investigations are in $G_n = G(V_n)$ when *n* is a small number (for example *n*≤3) we can use the *GrassmannAlgebra* package;
- if we want to do research in $G_n = G(V_n)$ when *n* is a large number it's better to use the second method. It's easier and faster to input Grassmann number, and the results are convenient for analysis;
- it is better to use this second method if we work with odd or even Grassmann numbers;
- the program in the second method multiplies numbers using only the wedge symbol "$\wedge$" until the Browne"s program uses the function *GrassmannExpandAndSimplify*.

Ultimately the both programs facilitate the work of those who are interested in Grassmann algebra.

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СРАВНЯВАНЕ НА ДВА МЕТОДА ЗА ПРЕСМЯТАНЕ С ГРАСМАНОВИ ЧИСЛА

Антоанета Михова

Русенски университет „Ангел Кънчев"

Резюме: *В статията са разгледани два метода за пресмятане с Грасманови числа. Описани са техните възможности и е направено сравнение между тях.*

Ключови думи: *Грасманова алгебра, Външно произведение, Пресмятане с Грасманови числа.*

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