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BOOK 5

"MATHEMATICS, INFORMATICS AND PHYSICS"

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BOOK 5

"MATHEMATICS, INFORMATICS AND PHYSICS"

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POLYNOMIAL IDENTITIES OF THE 2x2 MATRICES OVER THE FINITE DIMENSIONAL GRASSMANN ALGEBRA

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***Abstract**: The paper presents a survey of results obtained in the last years by different people in the field of matrix algebras over the Grassmann algebra. Some author's results concerning the Grassmann algebra and the matrix algebra over finite dimensional Grassmann algebra are presented. A conjecture in [8] appeared to be true and a proof of the corresponding theorem is given here.*

***Keywords**: Matrix algebras, Polynomial identities, Grassmann algebra.*

INTRODUCTION

Let K be a field of characteristic 0. A vector space R is called an algebra (or a $K-$algebra) if R is equipped with a binary operation "$*$" (i.e. a map $(R,R)\to R$), called a multiplication such that for any $a,b,c\in R$ and $\forall\alpha\in K$ $(a+b)*c=a*c+b*c$, $c*(a+b)=c*a+c*b$, $\alpha(a*b)=(\alpha a)*b=a*(\alpha b)$.

Usually we denote the multiplication of R by "." and write ab instead $a.b$. The notion of algebra generalizes both the notion of vector space and of ring. The algebra R is associative if $(a*b)*c=a*(b*c)$ for every $a,b,c\in R$, R is commutative if $a*b=b*a$, $a,b\in R$ and R is unitary if R has a unity e (i.e. $\exists\, e\in R:\forall a\in R\Rightarrow a*e=e*a=a$). For example $M_n(K)$- the set of all $n\times n$ matrices with entries from K is an associative, noncommutative, unitary algebra.

Let V be a vector space with ordered basis $\{e_{1,}\ e_2,...\}$. The Grassmann (or Exterior) algebra $G(V)$ of V is the associative algebra, generated by the basis of V with defining relations $e_ie_j+e_je_i=0$, for all $i,j=1,2,....$

The elements $e_i\in V,\ i=1,2,...$ are called generators of $G(V)$. Since $e_ie_j+e_je_i=0$ and $charK=0$ we have $e_i^2=0$.

The set $B=\{1\}\cup\{e_{i_1}e_{i_2}...e_{i_m}\,|\,1\le i_1<i_2<...<i_m, m=1,2,...\}$ is the basis of $G(V)$. If a is a basic element and $1\ne a=e_{i_1}e_{i_2}\dots e_{i_m}$, then m is called a length of a.

If V_n is a finite dimensional vector space with dimension n we denote by $G_n=G(V_n)$. $B_n=\{1,\ e_1,\ e_2,\ e_1e_2,\ e_3,\ e_1e_3,\ e_2e_3,\ e_1e_2e_3,...,e_1e_2...e_n\}$ is the basis of $G_n=G(V_n)$ and $\dim G_n=2^n$.

The expression $[x_1,x_2]=x_1x_2-x_2x_1$ is called commutator of x_1 and x_2.

The polynomial $s_n(x_1,x_2,...,x_n)=\sum_{\sigma\in S_n}sign(\sigma)\,x_{\sigma(1)}\,x_{\sigma(2)}...x_{\sigma(n)}$, where S_n is the symmetric group of degree n is called a standard polynomial.

The smallest degree of the multilinear identities, which an algebra *R* satisfies is called *PI-degree* of the algebra, denoted as $PI\deg(R)$.

POLYNOMIAL IDENTITIES

For the matrix algebra $M_n(K)$ basic results are the Cayley-Hamilton theorem and the famous Amitsur-Levitzki theorem. Amitsur and Levitzki [1] proved that in the matrix algebra $M_n(K)$ of order n over a field K the standard identity of degree $2n$ holds, i.e. $s_{2n}(x_1,x_2,...,x_{2n})=0.$

Krakowski and Regev proved in 1973 the following

Proposition 1 [6, Corollary, p.437] *The T-ideal T(G) is generated by the identity* $[[x_1,x_2],x_3]=[x_1,x_2,x_3]=0.$

For the algebra $G_n=G(V_n)$ holds

Proposition 2 [5, Exercise 5.3] *For* $G_n=G(V_n)$ *over* n*-dimensional vector space* V_n*,* $n>1$*, all identities follow from the identity* $[x_1,x_2,x_3]=0$ *and the standard identity* $s_{2p}(x_1,x_2,...,x_{2p})=0$*, where* p *is the minimal integer with* $2p>n$*.*

Berele and Regev proved

Proposition 3 [3, Lemma 6.1] *The algebra* $G(V)$ *satisfies the identity* $(s_n(x_1,x_2,...,x_n))^k=0$ *for all* $n,k>1$*.*

Proposition 4 [3, Lemma 3.2, p.123] *Let* R *be an algebra with* $PI\deg(R)=r$*, then* $PI\deg(M_n(R))\geq nr$*. In particular,* $PI\deg(M_n(G))\geq 3n$*.*

Proposition 5 [2, Lemma, p.1509] *The algebra* $M_n(G)$ *satisfies the identity* s_{2n}^k *for some* $k>1$ *but satisfies neither* s_{2n} *nor identities of the form* s_m^k *for any* k *when* $m<2n$*.*

A connection between the identities in $M_n(K)$ and $M_n(G)$ is given by M. Domokos and A. Popov.

Proposition 6 [4, Proposition 2.1, p.13] *Let* $f_1,\ldots,f_d\in K\langle x_1,\ldots x_m\rangle$ *be elements of the T-ideal of identities of* M_n*. If* $d>\frac{1}{2}n^2m$*, then* $f_1\ldots f_d=0$ *is an identity on* $M_n(G)$*.*

The above Proposition 6 has an analogue for the upper triangular matrices U_n [12].

A. Popov sets a more precise estimation of $PI\deg(M_n(G))$.

Proposition 7 [11, The main Theorem] *Let* $M_n(G)$ *be the matrix algebra of order* n *over the (infinite dimensional) Grassmann algebra. Then* $M_n(G)$ *has no identities of degree* $4n-2$*.*

USING *MATHEMATICA* TO OBTAIN IDENTITES IN THE MATRIX ALGEBRA OVER THE FINITE DIMENSIONAL GRASSMANN ALGEBRA

The elements of the Grassmann algebra are called *grassmann numbers*. A grassmann number containing only monomials with even length is called an *even grassmann number*, and a grassmann number containing only monomials with odd length is called an *odd grassmann number.*

Ts. Rashkova and A. Mihova find a correspondence [10] between the integers from 0 to 2^n-1 and the basic elements of the Grassmann algebra over a *n*-dimensional vector space.

Let i be an integer, $0 \le i \le 2^n - 1$ and $i = \alpha_1 \alpha_2 \dots \alpha_{n(2)}$ for $\alpha_i \in \{0,1\}, i = 1, 2, \dots, n$ be its binary representation. To each i is juxtaposed the basic element $e_1^{\alpha_n} e_2^{\alpha_{n-1}} \dots e_{n-1}^{\alpha_2} e_n^{\alpha_1}$.

Then any grassmann number $x \in G_n = G(V_n)$ can be express in the form:

$$x = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_1 e_2 + a_4 e_3 + a_5 e_1 e_3 + a_6 e_2 e_3 + a_7 e_1 e_2 e_3 + \dots + a_{2^n - 1} e_1 e_2 \dots e_n,$$

$a_i \in K, i = 0, 1, \dots, 2^n - 1.$

The formulated correspondence gives the possibility to work mainly with the indices of coefficients.

Using the above described correspondence two programs in *Mathematica* are done - one for multiplication of grassmann numbers [10] and another one for multiplication of 2×2 matrices with entries from a finite Grassmann algebra [9]. For a guide book in the system *Mathematica* we consider [15].

These two programs are used to verify identities related to the standard polynomial for the matrix algebra over finite dimensional Grassmann algebras. Some identities are verified for small *n*. Based on the obtained results the next propositions are formulated.

Let $M_k(G)$ be the matrix algebra of order k with entries from G. We denote by $M_2(G_n^0)$ the set of 2×2 matrices with entries even grassmann numbers and by $M_2(G_n^1)$ the set of 2×2 matrices with entries odd grassmann numbers from a finite Grassmann algebra G_n.

Proposition 9 [7, Proposition 5, p.19] $s_4(x_1, x_2, x_3, x_4) = 0$ *is an identity on* $M_2(G_n^0)$.

Proposition 10 [7, Proposition 6, p.19] $s_{n+1}(x_1, \dots, x_{n+1}) = 0$ *is an identity on* $M_2(G_n^1)$.

The following theorem was formulated as a conjecture in [8]. Here we give the proof.

Theorem 1 The algebra $M_2(G_n)$ satisfies the identity $s_4(x_1, x_2, x_3, x_4)^p = 0$, where p is the minimal integer with $2p > n$.

Proof. Let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix with entries from $G_n = G(V_n)$ and

$$a = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_1 e_2 + a_4 e_3 + \dots + a_{2^n - 1} e_1 e_2 \dots e_n,$$

$$b = b_0 + b_1 e_1 + b_2 e_2 + b_3 e_1 e_2 + b_4 e_3 + \dots + b_{2^n - 1} e_1 e_2 \dots e_n,$$

$$c = c_0 + c_1 e_1 + c_2 e_2 + c_3 e_1 e_2 + c_4 e_3 + \dots + c_{2^n - 1} e_1 e_2 \dots e_n,$$

$$d = d_0 + d_1 e_1 + d_2 e_2 + d_3 e_1 e_2 + d_4 e_3 + \dots + d_{2^n - 1} e_1 e_2 \dots e_n,$$

$a_i, b_i, c_i, d_i \in K, \; i = 0, 1, \dots, 2^n - 1.$

We can express X as follows

$$X = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} + \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} e_1 + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} e_2 + \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} e_1 e_2 +$$

$$+\begin{pmatrix} a_4 & b_4 \\ c_4 & d_4 \end{pmatrix} e_3 + \cdots + \begin{pmatrix} a_{2^{n-1}} & b_{2^{n-1}} \\ c_{2^{n-1}} & d_{2^{n-1}} \end{pmatrix} e_n + \cdots + \begin{pmatrix} a_{2^n-1} & b_{2^n-1} \\ c_{2^n-1} & d_{2^n-1} \end{pmatrix} e_1 e_2 \dots e_n.$$

Let denote the matrix $\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$ by X_i, $i = 0,1,\dots,2^n - 1$. Then

$$X = X_0 + X_1 e_1 + X_2 e_2 + X_3 e_1 e_2 + X_4 e_3 + \cdots + X_{2^{n-1}} e_n + \cdots + X_{2^n-1} e_1 e_2 \dots e_n.$$

We consider the matrices XJ and XJ_k for $J = 1, 2, 3, 4$ and $k = 1,\dots, 2^n - 1$, namely

$$XJ = XJ_0 + XJ_1 e_1 + XJ_2 e_2 + XJ_3 e_1 e_2 + XJ_4 e_3 + \cdots + XJ_{2^{n-1}} e_n + \cdots + XJ_{2^n-1} e_1 e_2 \dots e_n.$$

Using the multilinearity of the standard polynomial we transform it as:

$$\begin{aligned}
& s_4(X1, X2, X3, X4) = \\
& s_4(X1_0 + X1_1 e_1 + X1_2 e_2 + X1_3 e_1 e_2 + X1_4 e_3 + \cdots + X1_{2^n-1} e_1 e_2 \dots e_n, \\
& \quad X2_0 + X2_1 e_1 + X2_2 e_2 + X2_3 e_1 e_2 + X2_4 e_3 + \cdots + X2_{2^n-1} e_1 e_2 \dots e_n, \\
& \quad X3_0 + X3_1 e_1 + X3_2 e_2 + X3_3 e_1 e_2 + X3_4 e_3 + \cdots + X3_{2^n-1} e_1 e_2 \dots e_n, \\
& \quad X4_0 + X4_1 e_1 + X4_2 e_2 + X4_3 e_1 e_2 + X4_4 e_3 + \cdots + X4_{2^n-1} e_1 e_2 \dots e_n) = \\
& s_4(X1_0, X2_0, X3_0, X4_0) + \\
& + s_4(X1_1 e_1, X2_0, X3_0, X4_0) + s_4(X1_0, X2_1 e_1, X3_0, X4_0) + \\
& + s_4(X1_0, X2_0, X3_1 e_1, X4_0) + s_4(X1_0, X2_0, X3_0, X4_1 e_1) + \\
& + s_4(X1_2 e_2, X2_0, X3_0, X4_0) + s_4(X1_0, X2_2 e_2, X3_0, X4_0) + \\
& + s_4(X1_0, X2_0, X3_2 e_2, X4_0) + s_4(X1_0, X2_0, X3_0, X4_2 e_2) + \cdots \\
& \cdots + s_4(X1_{2^{n-1}} e_n, X2_0, X3_0, X4_0) + s_4(X1_0, X2_{2^{n-1}} e_n, X3_0, X4_0) + \\
& + s_4(X1_0, X2_0, X3_{2^{n-1}} e_n, X4_0) + s_4(X1_0, X2_0, X3_0, X4_{2^{n-1}} e_n) + S = \\
& s_4(X1_0, X2_0, X3_0, X4_0) + \\
& + s_4(X1_1, X2_0, X3_0, X4_0) e_1 + s_4(X1_0, X2_1, X3_0, X4_0) e_1 + \\
& + s_4(X1_0, X2_0, X3_1, X4_0) e_1 + s_4(X1_0, X2_0, X3_0, X4_1) e_1 + \\
& + s_4(X1_2, X2_0, X3_0, X4_0) e_2 + s_4(X1_0, X2_2, X3_0, X4_0) e_2 + \\
& + s_4(X1_0, X2_0, X3_2, X4_0) e_2 + s_4(X1_0, X2_0, X3_0, X4_2) e_2 + \cdots \\
& \cdots + s_4(X1_{2^{n-1}} X2_0, X3_0, X4_0) e_n + s_4(X1_0, X2_{2^{n-1}}, X3_0, X4_0) e_n + \\
& + s_4(X1_0, X2_0, X3_{2^{n-1}}, X4_0) e_n + s_4(X1_0, X2_0, X3_0, X4_{2^{n-1}}) e_n + S.
\end{aligned}$$

We denoted by S the sum of the other summands which are products of matrices with entries from K and the basic elements $e_1 e_2$, $e_1 e_3$, $e_2 e_3$, $e_1 e_2 e_3, \dots, e_1 e_2 \dots e_n$.

Since $XJ_0, XJ_1, XJ_2, \ldots, XJ_{2^n-1}$ for $J = 1, 2, 3, 4$ are matrices with entries from K, applying Amitsur-Levitzki theorem we obtain that

$$\begin{aligned}
&s_4(X1_0, X2_0, X3_0, X4_0) = s_4(X1_1, X2_0, X3_0, X4_0) = \\
&s_4(X1_0, X2_1, X3_0, X4_0) = s_4(X1_0, X2_0, X3_1, X4_0) = \\
&s_4(X1_0, X2_0, X3_0, X4_1) = s_4(X1_2, X2_0, X3_0, X4_0) = \\
&s_4(X1_0, X2_2, X3_0, X4_0) = s_4(X1_0, X2_0, X3_2, X4_0) = \\
&s_4(X1_0, X2_0, X3_0, X4_2) = \cdots = s_4\left(X1_{2^{n-1}}, X2_0, X3_0, X4_0\right) = \\
&s_4\left(X1_0, X2_{2^{n-1}}, X3_0, X4_0\right) = s_4\left(X1_0, X2_0, X3_{2^{n-1}}, X4_0\right) = \\
&s_4\left(X1_0, X2_0, X3_0, X4_{2^{n-1}}\right) = 0.
\end{aligned}$$

Hence $s_4(X1, X2, X3, X4) = S.$ Then for $2p > n$ we form $s_4^p = S^p$. As the summands of S are of length ≥ 2 then S^p is a sum of monomials with length $\geq 2p$ multiplied by matrices. Since $2p > n$ then each monomial in s_4^p contains at least one repeated generator. Hence $s_4^p = 0.$ This completes the proof of the theorem.

Vishne described in [14] an efficient way to use the $Sym(n)$ -module structure of the ideal of multilinear identities in the computation for a given algebra of such identities of degree n. The method was applied to be shown that the algebra $M_2(G)$ has identities of degree 8, but of no smaller degree.

An explicit form of the Vishne identities is given in [13] and identities are verified in $M_2(G_n)$ for small *n*.

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ПОЛИНОМНИ ТЪЖДЕСТВА В МАТРИЧНАТА АЛГЕБРА ОТ ВТОРИ РЕД НАД КРАЙНОМЕРНА ГРАСМАНОВА АЛГЕБРА

Антоанета Михова

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***Резюме**: В статията е направен обзор на резултатите, получени през последните години в областта на матричната алгебра над Грасманова алгебра. Представени са резултати, които се отнасят за Грасмановата алгебра и за матричната алгебра над крайномерна Грасманова алгебра. Доказана е теорема, която е формулирана като хипотеза в [8].*

***Ключови думи**: Матрична алгебра, Полиномни тъждества, Грасманова алгебра.*

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