PROCEEDINGS

of the Union of Scientists - Ruse

Book 5

Mathematics, Informatics and Physics

Volume 13, 2016

RUSE

PROCEEDINGS
OF THE UNION OF SCIENTISTS - RUSE

Union of Scientists - Ruse
16, Konstantin Irechek Street
7000 Ruse
BULGARIA
Phone: (++359 82) 828 135,
(++359 82) 841 634
E-mail: suruse@uni-ruse.bg
web: suruse.uni-ruse.bg

Contacts with Editor
Phone: (++359 82) 888 738
E-mail: zzdravkova@uni-ruse.bg

PROCEEDINGS
of the Union of Scientists – Ruse

ISSN 1314-3077

The Ruse Branch of the Union of Scientists in Bulgaria was founded in 1956. Its first Chairman was Prof. Stoyan Petrov. He was followed by Prof. Trifon Georgiev, Prof. Kolyo Vasilev, Prof. Georgi Popov, Prof. Mityo Kanev, Assoc. Prof. Boris Borisov, Prof. Emil Marinov, Prof. Hristo Beloev. The individual members number nearly 300 recognized scientists from Ruse, organized in 13 scientific sections. There are several collective members too – organizations and companies from Ruse, known for their success in the field of science and higher education, or their applied research activities. The activities of the Union of Scientists – Ruse are numerous: scientific, educational and other humanitarian events directly related to hot issues in the development of Ruse region, including its infrastructure, environment, history and future development; commitment to the development of the scientific organizations in Ruse, the professional development and growth of the scientists and the protection of their individual rights.

The Union of Scientists – Ruse (US – Ruse) organizes publishing of scientific and popular informative literature, and since 1998 – the “Proceedings of the Union of Scientists- Ruse".

BOOK 5

"MATHEMATICS, INFORMATICS AND PHYSICS"

VOLUME 13

CONTENTS

Mathematics

Informatics

BOOK 5

"MATHEMATICS, INFORMATICS AND PHYSICS"

VOLUME 13

Physics

web: suruse.uni-ruse.bg

EXISTENCE OF SOLUTIONS OF SECOND ORDER NONLINEAR DIFFERENCE PROBLEMS

Nikolay Dimitrov

Angel Kanchev University of Ruse

Abstract: *This paper is devoted to the study of second order nonlinear difference equations. A nonlocal perturbation of a Dirichlet boundary value problem is considered. An exhaustive study of the related Green's function to the linear part is done. The exact expression of the function is given, moreover the range of parameter for which it has a constant sign is obtained. Using this, some existence results for the nonlinear problem are deduced from the classical Krasnoselski fixed point theorem.*

Keywords: *Difference equations, Green's function, Parameter dependence, Fixed point theorems.*

INTRODUCTION AND PRELIMINARIES

The study of difference equations represents an useful model for several mathematical models in different areas as economy or population dynamics. Moreover they are a fundamental tool in the discretization of a differential equation. The reader can found the classical theory on the monographs of R. P. Agarwal [1], S. Elaydi [8], Goldberg [10], V. Laksmikantham and D. Trigiante [13].

The existence of solution for nonlinear difference problems has been treated in different ways . Here we mention the method of upper and lower solutions [2, 9], monotone iterative techniques [5, 16], variational methods [6, 7, 14] or fixed point theorems [3, 15].

Our aim is to obtain the existence result for the following parameter dependence nonlinear problem

$$(P)\begin{cases} -\Delta^2 u(k-1) = f(k,u(k)), \quad k = 1,\dots,N-1, \quad N \ge 2, \\ u(0)=0, \quad u(N) = \lambda \sum_{k=1}^{N-1} u(k), \quad \lambda > 0, \end{cases}$$

which is a nonlocal perturbation of the Dirichlet problem.

In the next Section 2 we study the Green's function related to considering the linear problem. The exact expression of such function is obtained. Moreover, we found the values of the parameter λ for which this function is strictly positive on its domain. The last Section is devoted to the study of the existence of solution for the considered nonlinear boundary value problem by using the classical expansion/contraction Krasnoselskii's fixed point theorem. An example is given.

STUDY OF THE GREEN'S FUNCTION

Denote $I = \{0,\dots,N\}$, $J = \{1,\dots,N-1\}$ and $J_1 = \{1,\dots,N\}$.

The Green's function to the related linear problem

$$\begin{cases} -\Delta^2 u(k-1) = 0, \quad k \in J, \quad N \ge 2, \\ u(0)=0, \quad u(N) = \lambda \sum_{k=1}^{N-1} u(k), \quad \lambda > 0, \end{cases}$$

is given by a two variables function

$$G:(k,s)\in I\times J\to G(k,s)\in \mathbb{R}.$$

The main result of this section is the following

Theorem 1. *If* $\lambda\in\left(0,\dfrac{2}{N-1}\right)$ *then* $G(k,s)>0$ *for all* $k\in J_1$ *and* $s\in J$ *.*

Moreover, for such values of λ, there are two positive constants $0<m<M$ for which

$$mG(N,s)\le G(k,s)\le MG(N,s) \tag{1}$$

for all $k\in J_1$ and $s\in J$.

Proof: Following the approach in [12], we know that function G follows a form as

$$G(k,s)=\begin{cases}G_1(k,s)\equiv C_1(s)+kC_2(s), & \text{if} k\le s,\\ G_2(k,s)\equiv C_3(s)+kC_4(s), & \text{if} k>s.\end{cases}$$

To obtain the expression of the functions C_i , $i\in\{1,\ldots,4\}$, we must use the fact that the expression of the Green's function G is characterized by the following identities:

$$-\Delta^2 G(k-1,k)=1,$$

$$\Delta^2 G(k-1,s)=0,\quad s\ne k,$$

$$G(0,s)=0,\quad G(N,s)=\lambda\sum_{k=1}^{N-1}G(k,s).$$

The diagonal equalities give us that

$$\begin{aligned}-1&=G(k+1,k)-2G(k,k)+G(k-1,k)\\ &=-(k+1)C_2(k)+C_3(k)+(k+1)C_4(k)\end{aligned}$$

and

$$\begin{aligned}0&=G(k+2,k)-2G(k+1,k)+G(k,k)\\ &=kC_2(k)-C_3(k)-kC_4(k).\end{aligned}$$

Moreover, the condition $G(0,s)=0$ implies that $C_1(s)=0.$

We may write the last condition as

$$G(N,s)=\lambda\sum_{k=1}^{N-1}G(k,s)=\lambda\sum_{k=1}^{s}G_1(k,s)+\lambda\sum_{k=s+1}^{N-1}G_2(k,s).$$

Thus the last equality about Green's function is as follows:

$$\begin{aligned}G(N,s)&=\lambda\sum_{k=1}^{N-1}G(k,s)=\lambda\sum_{k=1}^{s}kC_2(s)+\lambda\sum_{k=s+1}^{N-1}\left(C_3(s)+kC_4(s)\right)\\ &=\lambda\frac{s(s+1)}{2}C_2(s)+\lambda(N-s-1)C_3(s)+\lambda\frac{(N+s)(N-s-1)}{2}C_4(s)\\ &=C_3(s)+NC_4(s).\end{aligned}$$

After solving the system we obtain that

$$C_2(s)=\frac{(N-s)\left(2-\lambda(N-s-1)\right)}{N\left(2-\lambda(N-1)\right)},$$

$$C_3(s)=s$$

and

$$C_4(s)=\frac{s(-2+\lambda(2N-s-1))}{N\left(2-\lambda(N-1)\right)}.$$

Finally for $\lambda\ne\dfrac{2}{N-1}$ the related Green's function to our problem is given by the following expression:

$$G(k,s)=\begin{cases}\dfrac{k(N-s)\big(2-\lambda(N-s-1)\big)}{N\big(2-\lambda(N-1)\big)}, & k\le s,\\[2mm] s+\dfrac{ks(-2+\lambda(2N-s-1))}{N\big(2-\lambda(N-1)\big)}, & k>s.\end{cases}$$

One can check that $G_1(k,s)>0$ when $\lambda\in\left(0,\dfrac{2}{N-1}\right)$.

Now we will show that $G_2(k,s)>0$ for all $\lambda\in\left(0,\dfrac{2}{N-1}\right)$. From the expression of Green's function we have that $G_2(k,s)>0$ whenever

$$sN\big(2-\lambda(N-1)\big)+ks(-2+\lambda(2N-s-1))>0,$$

which is the same as

$$2(N-k)>\lambda\big(N^2-N-2Nk+sk+k\big).$$

Since $\lambda<\dfrac{2}{N-1}$ it is enough to show that

$$2(N-k)>\frac{2\big(N^2-N-2Nk+sk+k\big)}{N-1},$$

which is true since $N>s$.

In the sequel, we are interested in to find two positive constants m and M, satisfying inequalities (1) for $1\le s\le N-1$.

Let us consider the case when $k\le s$. Then

$$\frac{G_1(k,s)}{G_2(N,s)}=\frac{k\big(2-\lambda(N-s-1)\big)}{\lambda sN}.$$

It is clear that

$$\frac{\min\{k\big(2-\lambda(N-s-1)\big)\}}{\max\{\lambda sN\}}\le\frac{G_1(k,s)}{G_2(N,s)}\le\frac{\max\{k\big(2-\lambda(N-s-1)\big)\}}{\min\{\lambda sN\}}.$$

Thus we obtain that

$$\frac{k\big(2-\lambda(N-2)\big)}{\lambda N(N-1)}\le\frac{G_1(k,s)}{G_2(N,s)}\le\frac{2k}{\lambda N}.$$

Now, let us consider the case when $k>s$. We have that

$$\frac{G_2(k,s)}{G_2(N,s)}=\frac{2(N-k)-\lambda\big(N^2-N-2kN+k+ks\big)}{\lambda N(N-s)}.$$

Using the same arguments as before, we obtain that

$$\frac{2(N-k)-\lambda N\big(N-k-1\big)}{\lambda N(N-1)}\le\frac{G_2(k,s)}{G_2(N,s)}\le\frac{2(N-k)-\lambda\big(N^2-N-2kN+2k\big)}{\lambda N}.$$

Let us denote

$$m_1(k)=\frac{k\big(2-\lambda(N-2)\big)}{\lambda N(N-1)},\quad m_2(k)=\frac{2(N-k)-\lambda N\big(N-k-1\big)}{\lambda N(N-1)},$$

$$M_1(k)=\frac{2k}{\lambda N}\quad\text{and}\quad M_2(k)=\frac{2(N-k)-\lambda\big(N^2-N-2kN+2k\big)}{\lambda N}.$$

Finally, we conclude that, by defining $m=\min\{m_1,m_2\}$ and $M=\max\{M_1,M_2\}$, then for all $\lambda\in\left(0,\dfrac{2}{N-1}\right)$ the inequalities (1) hold for our problem.

KRASNOSELSKII'S FIXED POINT THEOREM

In this chapter we will prove the existence of solutions of problem (P), using the results proved in the previous section about the related Green's function and the classical expansion/contraction Krasnoselskii's fixed point theorem.

We now recall some definitions that will be useful in the sequel: a subset K of a real Banach space N is a *cone* if and only if it is closed, $K+K\subset K$, $\lambda K\subset K$ for all $\lambda\geq 0$ and $K\cap(-K)=\{0\}$.

In the sequel we enunciate the classical expansion/contraction Krasnoselskii's fixed point Theorem (see [17, Theorem 13.D]).

Theorem 2. *Let $T:K\to K$ be a completely continuous operator and $0<r<R$. Moreover, if one of the following conditions are fulfilled:*

(i) $\|Tu\|\leq\|u\|$ for any $u\in K$ with $\|u\|=r$ and $\|Tu\|\geq\|u\|$ for any $u\in K$ with $\|u\|=R$, or

(ii) $\|Tu\|\geq\|u\|$ for any $u\in K$ with $\|u\|=r$ and $\|Tu\|\leq\|u\|$ for any $u\in K$ with $\|u\|=R$,

then operator T has a fixed point in K such that $r<\|x\|<R$.

Let us define the operator

$$Tu(k):=\sum_{s=1}^{N-1}G(k,s)f(s,u(s)),\quad k\in I, \tag{2}$$

and a cone

$$K=\left\{u:I\to[0,\infty),\, u(k)\geq\frac{m}{M}\|u\|, k\in J_1\right\}, \tag{3}$$

where

$$\|u\|:=\max\{|u(k)|,\quad k\in I\}.$$

To the end of this chapter we assume the following condition:

(F) $f:J\times[0,\infty)\to[0,\infty)$ is a continuous function.

Let us introduce the following notations that will be used along the paper:

$$f_0^-=\lim_{u\to 0^+}\min_{s\in J}\frac{f(s,u)}{u},\quad f_0^+=\lim_{u\to 0^+}\max_{s\in J}\frac{f(s,u)}{u},$$

$$f_\infty^-=\lim_{u\to\infty}\min_{s\in J}\frac{f(s,u)}{u}\quad\text{and}\quad f_\infty^+=\lim_{u\to\infty}\max_{s\in J}\frac{f(s,u)}{u}.$$

Now, in order to deduce existence results for problem (P), we introduce the following conditions:

(H1) $\lambda\in\left(0,\dfrac{2}{N-1}\right)$.

(H2) $f_0^-=\infty$ and $f_\infty^+=0$.

(H3) $f_0^+=0$ and $f_\infty^-=\infty$.

Let $u\in K$ be arbitrarily chosen, then, by assuming conditions (F) and $(H1)$ we have that $Tu\geq 0$ on I. Moreover, from (1), we deduce that the following inequalities are fulfilled for all $k\in J_1$:

$$\begin{aligned}Tu(k)&=\sum_{s=1}^{N-1}G(k,s)f(s,u(s))\geq m\sum_{s=1}^{N-1}G(N,s)f(s,u(s))\\&\geq\frac{m}{M}\sum_{s=1}^{N-1}\max_{k\in I}\{G(k,s)\}f(s,u(s))\geq\frac{m}{M}\max_{k\in I}\left\{\sum_{s=1}^{N-1}G(k,s)f(s,u(s))\right\}\\&=\frac{m}{M}\|Tu\|.\end{aligned}$$

In other words, $T: K \to K.$

Moreover, since f is continuous, it is clear that T is completely continuous. So we are in the hypothesis of Krasnoselskii's fixed point Theorem 2.

Our main result of this paper is the following.

Theorem 3. *Suppose that conditions (F), (H1) and either (H2) or (H3) are satisfied. Then problem (P) has a positive solution on* J_1*.*

Proof: First, notice that the solutions of problem (P) coincide with the fixed points of operator T .

Assuming, at the beginning, that condition $(H2)$ is fulfilled.

Since $f_0^- = \infty,$ for $\delta_1 \geq \dfrac{1}{\frac{m}{M}\max\limits_{k\in I}\left\{\sum_{s=1}^{N-1}G(k,s)\right\}} > 0$ there exists $r > 0,$ such that $f(s,u) \geq \delta_1 u$ for all $0 < u \leq r$ and $s \in J.$

Let $u \in K, \|u\| = r$ then

$$\|Tu\| = \max_{k\in I}\left\{\sum_{s=1}^{N-1}G(k,s)\,f(s,u(s))\right\} \geq \delta_1 \max_{k\in I}\left\{\sum_{s=1}^{N-1}G(k,s)u(s)\right\}$$

$$\geq \delta_1\|u\|\frac{m}{M}\max_{k\in I}\left\{\sum_{s=1}^{N-1}G(k,s)\right\} \geq \|u\|.$$

Since $f_\infty^+ = 0$, for $0 < \delta_2 \leq \dfrac{1}{\max\limits_{k\in I}\left\{\sum_{s=1}^{N-1}G(k,s)\right\}}$, we can choose $R > r > 0,$ such that $f(s,u) \leq \delta_2 u$ for all $u \geq R$ and $s \in J.$

Let $u \in K, \|u\| = \dfrac{M}{m}R$ then $u(k) \geq \dfrac{m}{M}\|u\| = R$, for all $k \in J$ and

$$\|Tu\| = \max_{k\in I}\left\{\sum_{s=1}^{N-1}G(k,s)\,f(s,u(s))\right\} \leq \delta_2 \max_{k\in I}\left\{\sum_{s=1}^{N-1}G(k,s)u(s)\right\}$$

$$\leq \delta_2\|u\|\max_{k\in I}\left\{\sum_{s=1}^{N-1}G(k,s)\right\} \leq \|u\|.$$

If condition $(H3)$ is fulfilled, the proof is analogous.

In both situations, the existence of a positive solution on J_1 holds from Theorem 2 and the fact that the fixed points of operator T coincide with the solutions of problem (P).

Now we give an example to illustrate the main result.

Example 4. *Consider the following problem*

$$\begin{cases} -\Delta^2 u(k-1) + g(k)u^\beta(k) = 0, \quad k \in J, \quad N \geq 2, \quad \beta > 0 \\ u(0) = 0, \quad u(N) = \dfrac{\alpha}{N-1}\displaystyle\sum_{k=1}^{N-1}u(k), \quad 0 < \alpha < 2, \end{cases}$$

with $g(k) > 0$ for all $k \in J$.

It is obvious that conditions (F) and $(H1)$ are satisfied.

Furthermore, condition $(H3)$ is fulfilled for all $\beta > 1$, and condition $(H2)$ holds for any $\beta \in [0,1)$. So in both situations we are in the conditions of Theorem 3 and we can ensure that this problem has a positive solution on J_1.

When $\beta = 1$, our problem is linear and, in the particular case of $g(k) = 1$ for all $k \in J$, has as unique solution $u \equiv 0$. We point out that in this case neither $(H2)$ nor $(H3)$ are fulfilled. So this problem shows us that the conditions imposed in Theorem 3 are, in some sense optimal.

ACKNOWLEDGEMENT
This work is supported by the fund "Science research" under Project 16-FNSE-03.

REFERENCES

[1] R. P. Agarwal, Difference equations and inequalities. Theory, methods, and applications. Second edition. *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker Inc., New York, 2000.

[2] C. Bereanu and J. Mawhin, Existence and multiplicity results for nonlinear second order difference equations with Dirichlet boundary conditions. *Math. Bohem.* 131 (2006), 145–160.

[3] A. Cabada, N. Dimitrov, Multiplicity results for nonlinear periodic fourth order difference equations with parameter dependence and singularities. J. Math. Anal. Appl. 371 (2010), 2, 518–533.

[4] A. Cabada, A. Iannizzotto and S. Tersian, *Multiple solutions for discrete boundary value problems*, J. Math. Anal. Appl. 356 (2009) 418 – 428.

[5] A. Cabada and V. Otero-Espinar, Optimal existence results for nth order periodic boundary value difference equations. *J. Math. Anal. Appl.* 247 (2000), 67–86.

[6] A. Cabada, D. Souroujon, S. Tersian, Heteroclinic solutions of a second-order difference equation related to the Fisher-Kolmogorov's equation. Appl. Math. Comput. 218 (2012), 18, 9442–9450.

[7] X. Cai and Z. Guo, Existence of solutions of nonlinear fourth order discrete boundary value problem. *J. Difference Equ. Appl.* 12 (2006), 459–466.

[8] S. N. Elaydi, An introduction to difference equations, Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1996.

[9] D. Franco, D. O'Regan and J. Perán, Upper and lower solution theory for first and second order difference equations. *Dynam. Systems Appl.* 13 (2004), 273–282.

[10] S. Goldberg, Introduction to difference equations, New York, John Wiley & sons, 1960.

[11] M. A. Krasnoselskii, *Positive Solutions of Operator Equations*, Noordhoff, Groningen, 1964.

[12] W. Kelley, A. Peterson, Difference Equations, An Introduction with Applications, Academic Press, New York (1991).

[13] V. Lakshmikantham and D. Trigiante, Theory of difference equations. Numerical methods and applications, Mathematics in Science and Engineering, 181. Academic Press, Inc., Boston, MA, 1988.

[14] X. Liu, Y. Zhang and H. Shi, Existence of periodic solutions for a class of nonlinear difference equations. Qual. Theory Dyn. Syst. 14 (2015), 1, 51–69.

[15] J. Reunsumrit and T. Sitthiwirattham, Positive solutions of differencesummation boundary value problem for a second-order difference equation. J. Comput. Anal. Appl. 19 (2015), 2, 319–334.

[16] Y. M. Wang, Monotone methods for a boundary value problem of second-order discrete equation. *Comput. Math. Appl.* 36 (1998), 77–92.

[17] E. Zeidler, *Nonlinear functional analysis and its applications. I. Fixed-point theorems*, Springer-Verlag, New York (1986).

CONTACT ADDRESS
Assist. Nikolay Dimitrov
Department of Mathematics
Faculty of Natural Sciences and Education
Angel Kanchev University of Ruse
8 Studentska Str.,
7017 Ruse, Bulgaria
E-mail: ndimitrov@uni-ruse.bg

СЪЩЕСТВУВАНЕ НА МНОГО РЕШЕНИЯ НА ГРАНИЧНА ЗАДАЧА ЗА ДИСКРЕТНО *P*-ЛАПЛАСОВО УРАВНЕНИЕ ОТ ЧЕТВЪРТИ РЕД

Николай Димитров

Русенски университет "Ангел Кънчев"

Резюме: *В настоящата работа се доказва съществуване на поне три решения на нелинейно p-Лапласово диференчно уравнение от четвърти ред. Аргументите се базират на теоремата за три критични точки на Ричери. Даден е пример, илюстриращ получените резултати.*
Ключови думи: *Диференчни уравнения, p-Лапласиан, вариационни методи.*

СЪЩЕСТВУВАНЕ НА РЕШЕНИЕ НА НЕЛИНЕЙНА ДИФЕРЕНЧНА ЗАДАЧА ОТ ВТОРИ РЕД

Николай Димитров

Русенски университет "Ангел Кънчев"

Резюме: *В настоящата работа се доказва съществуване на решение на нелинейна диферечна задача от втори ред, зависеща от параметър. Изследвана е функцияна на Грийн за линейната задача в зависимост от стойностите на параметъра и е приложена класическата теорема на Красноселски за неподвижната точка в конуси. Даден е пример, илюстриращ получените резултати.*
Ключови думи: *Диференчни уравнения, функция на Грийн, неподвижни точки.*

ISSN 1314-3077

9 771314 307000