# PROCEEDINGS

## of the Union of Scientists - Ruse

Book 5

## Mathematics, Informatics and Physics

Volume 13, 2016

RUSE

# PROCEEDINGS
OF THE UNION OF SCIENTISTS - RUSE



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**of the Union of Scientists – Ruse**

**ISSN 1314-3077**

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**BOOK 5**

**"MATHEMATICS, INFORMATICS AND PHYSICS"**

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# HETEROCLINIC SOLUTIONS ON A SECOND-ORDER DIFFERENCE EQUATION

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***Abstract:*** *We study the existence of heteroclinic solutions for semilinear second-order difference equations related to the Fisher-Kolmogorov's equation* $\Delta^2 y(t-1)+k\Delta y(t-1)+f(y(t))=0$ *for* $k\in(1,2)$. *Analogous equation is considered in* [5] *and this paper continues the considerations there. The proves of the present results are based on monotonicity and continuity arguments.*

***Keywords****: Heteroclinic solutions, Initial value problem, Difference equations*

## INTRODUCTION

In the present paper we study the existence of heteroclinic solutions of the second-order difference equation

$$\Delta^2 y(t-1)+k\Delta y(t-1)+f(y(t))=0,\ t\in Z \ \text{ for } \ k\in(1,\ 2) \qquad (1)$$

under suitable assumptions, given in the next Section. This paper extends the consideration for the case $k\in(0,1)$. The proof of the presented results is based on monotonicity and continuity arguments. Equation (1) is related to Fisher-Kolmogorov's equation $u_t = u_{xx}+g(u)$, which was introduced in the papers of Fisher [7] and Kolmogorov [8] and it is originally motivated by models in population dynamic. Looking for traveling waves $u(x,t)=U(x-Ct)$, with speed $C$, one obtains the second-order ODE

$$U''+CU'+g(U)=0. \qquad (2)$$

We note that a similar equation

$$\Delta^2 y(t-1)+c\Delta y(t)+f(y(t))=0,\ t\in Z \qquad (3)$$

is considered in [5]. It is easy to see that (3) is equivalent to Equation (1) with $k=\frac{c}{c+1}$. In [5] is considered the case when $c>0$, i.e. $k\in(0,1)$. In the present paper is considered the case $k\in(1,2)$ under some additional conditions for the function $f(.)$, given in the next Section. Here we derive our main results using simple monotonicity and continuity arguments. As it is described in [1], Equations (1) and (3) appear after a discretization and rescaling of Eq (2). Fast and heteroclinic solutions of Eq (2) are studied in the paper of Arias [2]. Several metods of considerations of various classes of difference equations can be found for example in [3], [4], [6], [9] etc.

## BASIC ASSUMPTIONS AND THE BEHAVIOR OF SOLUTIONS FOR $t\geq 0$.

We consider the equation (1), where $\Delta y(t)=y(t+1)-y(t)$, $\Delta^2 y(t)=y(t+2)-2y(t+1)+y(t)$, $Z$ is the set of integers and the constant $k\in(1,2)$. For the function $f(.)$ we suppose that the following conditions are fulfilled:

C1. For any two numbers $x\in[0,1]$ and $y\in[0,1]$ and $x\neq y$,

$$|f(x)-f(y)|<(2-k)\ |x-y|; \qquad (4)$$

C2. $f:[0,1]\to R_+$ , where $f(0)=f(1)=0$ and $f(y)>0$ for $y\in(0,1)$;

C3. $f(x)<(k-1)(1-x)$ and $f$ is strictly monotonous decreasing function in a small left neighbourhood of $1$.

Obviously we can define the function $f(y)$ so that it is defined $\forall y\in R$ , as $f(y)=0$ for $y\notin[0,1]$. So the function $f(.)$ such defined, satisfies Lipshich condition for any two numbers $x\in R$ and $y\in R$ . In particular thus defined function $f(y)$ is continuous $\forall y\in R$ .

**Lemma 1.** *Let the function $f$ satisfy the assumptions C1, C2 and C3. Let $l$ be an arbitrary positive integer and $z$ is an arbitrary real number, such that $z\in(0,1)$ . Then there exists a real number $z_0\in[z,1)$ such that there exists a solution $y(t)$ of (1), satisfying the conditions: $y(t_0)=z_0$ , $y(t_0+l)=z$ , ( $t_0$ is an arbitrary integer ), $\{y(t)\}$ is a monotonous decreasing function for $t\geq t_0$ , i.e. $y(t+1)<y(t)$ for $t\geq t_0$ , $t\in Z$ , $y(t)>0$ for $t\geq t_0$ , $t\in Z$ and $\lim y(t)=0$ for $t\to+\infty$ .*

**Proof:** Equation (1) can be written in the form

$$y(t+1)-y(t)=-(k-1)(y(t)-y(t-1))-f(y(t))=(1-k)(y(t)-y(t-1))-f(y(t)). \qquad (5)$$

If in (5) we replace $t$ whith $t-1$ , we obtain

$$y(t)-y(t-1)=(1-k)(y(t-1)-y(t-2))-f(y(t-1)) \qquad (6)$$

and now from (5) we substract (6):

$$(y(t+1)-y(t))-(y(t)-y(t-1))$$
$$=(1-k)[(y(t)-y(t-1))-(y(t-1)-y(t-2))]-(f(y(t))-f(y(t-1)));$$
$$y(t+1)-y(t)=(2-k)(y(t)-y(t-1))-(1-k)(y(t-1)-y(t-2))-(f(y(t))-f(y(t-1)))\ ;$$
$$y(t+1)-y(t)=[(2-k)(y(t)-y(t-1))-(f(y(t))-f(y(t-1)))]+(k-1)(y(t-1)-y(t-2)). \qquad (7)$$

From condition C1, the sign of $(2-k)(y(t)-y(t-1))-(f(y(t))-f(y(t-1)))$ of (7) coinsides with the sign of $y(t)-y(t-1)$ for $k<2$ . Thus from (7) it follows that for $k\in(1,2)$,

if $y(t-1)-y(t-2)\geq 0$ and $y(t)-y(t-1)\geq 0$ , then $y(t+1)-y(t)\geq 0$ ;

and

if $y(t-1)-y(t-2)\leq 0$ and $y(t)-y(t-1)\leq 0$ , then $y(t+1)-y(t)\leq 0$ .

And also if at least one of the first or the second inequality is strong, then the thirth inequality is also strong.

So it follows that if we choose the values $y(t_0)\in(0,1)$ and $y(t_0+1)\in(0,1)$ such that

$$y(t_0)>y(t_0+1)>y(t_0+2), \qquad (8)$$

then $y(t_0+1)-y(t_0)<0$ and $y(t_0+2)-y(t_0+1)<0$ and it follows that $y(t_0+3)-y(t_0+2)<0$, i.e. $y(t_0)>y(t_0+1)>y(t_0+2)>y(t_0+3)$ . By analogy $y(t_0+3)>y(t_0+4)$ etc.

Thus we obtain that $y(t+1)<y(t)$ for $t=t_0$ , $t_0+1$ , $t_0+2$ , ..., i.e. the solution $\{y(t)\}$ of (1) is strictly monotonous decreasing function for $t\geq t_0$ , $t\in Z$ . So we have to choose the values $y(t_0)$ and $y(t_0+1)$ of the interval $(0,1)$ such that (8) is valid and

$y(t_0+2)$ is obtained from (5). Let now $y(t_0)=1-\varepsilon_1$ , $y(t_0+1)=1-\varepsilon_1-\varepsilon_2$ , where $\varepsilon_1>0$ and $\varepsilon_2>0$ are sufficiently small numbers. From (5)

$$y(t_0+2)-y(t_0+1)=-(k-1)(y(t_0+1)-y(t_0))-f(y(t_0+1))$$
$$=-(k-1)(-\varepsilon_2)-f(1-\varepsilon_1-\varepsilon_2),$$

i.e.

$$\mathrm{y}(\mathrm{t}_0+2)-\mathrm{y}(\mathrm{t}_0+1)=(\mathrm{k}-1)\ \varepsilon_2-\mathrm{f}(1-\varepsilon_1-\varepsilon_2). \tag{9}$$

At first we choose $\varepsilon_2>0$ sufficiently small and then $\varepsilon_1>0$ be such that the right hand side of (9) is less than $0$. It is posibile because $f(y)>0$ for $y\in(0,1)$. It shows that we can choose the small numbers $\varepsilon_1>0$ and $\varepsilon_2>0$ such that (9) is negative, i.e. $y(t_0+2)-y(t_0+1)<0$ , which provides that $\{y(t)\}$ is monotonous decreasing for $t\geq t_0$ .

From (5), $y(t+1)=(2-k)y(t)+(k-1)y(t-1)-f(y(t))\geq(k-1)y(t-1)>0$ if $y(t)>0$ and $y(t-1)>0$ , since from C2 $|f(y(t))|=f(y(t))<(2-k)y(t)$ for $y(t)>0$ . Thus we obtain that if $y(t_0)>0$ and $y(t_0+1)>0$ , then $y(t_0+2)>0$ , $y(t_0+3)>0$ , ..., i.e.

$y(t)>0$ for $t=t_0$ , $t_0+1$ , $t_0+2$ , ... .

Obviously this monotonous decreasing sequence with positive members has finite limit $l=\lim\limits_{t\to+\infty}y(t)\in[0,1)$ . If in (5) we take the limit for $t\to+\infty$ we obtain $f(l)=0$. Since $f(y)>0$ for $y\in(0,1)$, it follows that $l=0$, i.e. the monotonous decreasing sequence $\{y(t)\}_{t=t_0}^{+\infty}$ , for which $y(t)\in(0,1)$ for $t\geq t_0$ , $t\in \mathrm{Z}$ , tends to $0$ . Now we show that we can choose $y(t_0)$ and $y(t_0+1)$ of the interval $(0,1)$ such that for the corresponding monotonous decreasing sequence $\{y(t)\}_{t=t_0}^{+\infty}$ satisfying (5), the numbers $y(t)$, $t=t_0$ , $t_0+1$ , ..., $t_0+l$ , are sufficiently close to $1$.

At first we note that from (7) and condition C1,

$$|\ y(t+1)-y(t)|\leq(4-k)|\ y(t)-y(t-1)|+(k-1)|\ y(t-1)-y(t-2)|$$
$$\leq((4-k)+(k-1))\max\{|\ y(t)-y(t-1)|,|\ y(t-1)-y(t-2)|\},$$

i.e.

$$|\ y(t+1)-y(t)|\leq3\max\{|\ y(t)-y(t-1)|,|\ y(t-1)-y(t-2)|\}. \tag{10}$$

If the sequence $\{y(t)\}_{t=t_0}^{+\infty}$ is decreasing, then

$$0<\mathrm{y}(\mathrm{t})-\mathrm{y}(\mathrm{t}+1)\leq3\max\{\ \mathrm{y}(\mathrm{t}-1)-\mathrm{y}(\mathrm{t}),\ \mathrm{y}(\mathrm{t}-2)-\mathrm{y}(\mathrm{t}-1)\}. \tag{11}$$

If $y(t_0)=1-\varepsilon_1$ , $y(t_0+1)=1-\varepsilon_1-\varepsilon_2$ , then $y(t_0)-y(t_0+1)=\varepsilon_2>0$ and from (9) $y(t_0+1)-y(t_0+2)=f(1-\varepsilon_1-\varepsilon_2)-(k-1)\varepsilon_2$ . We choose sufficiently small numbers $\varepsilon_1>0$ and $\varepsilon_2>0$ such that the number $f(1-\varepsilon_1-\varepsilon_2)-(k-1)\varepsilon_2>0$ is sufficiently small. It is possibile, because $f(y)>0$ for $y\in(0,1)$ , $f(1)=0$ and $f(y)$ is continuous function. Denote $\varepsilon_0=\max\{\varepsilon_2, f(1-\varepsilon_1-\varepsilon_2)-(k-1)\varepsilon_2\}>0$ , which can be an sufficiently small positive number for appropriate choice of $\varepsilon_1$ and $\varepsilon_2$ and

$\varepsilon_0=\max\{y(t_0)-y(t_0+1),y(t_0+1)-y(t_0+2)\}$. Then from (11) $0<y(t_0+s)-y(t_0+s+1)<3^s\varepsilon_0$ for $s=0,1$ , ..., $l-1$ and hence

$$y(t_0+l)=y(t_0)-(y(t_0)-y(t_0+1))-(y(t_0+1)-y(t_0+2))-...-(y(t_0+l-1)-y(t_0+l))$$
$$>y(t_0)-(1+3+3^2+...+3^{l-1})\varepsilon_0=y(t_0)-\tfrac{3^l-1}{3-1}\varepsilon_0,$$

i.e. $y(t_0)-\frac{3^l-1}{2}\varepsilon_0 < y(t_0+l) < y(t_0)$ . Since $l \in \mathrm{N}$ is a fixed positive integer, then for any given in advance number $z$ , we can choose $\varepsilon_0 > 0$ sufficiently small and such that

$$y(t_0)-\frac{3^l-1}{2}\varepsilon_0 = 1-\varepsilon_1-\frac{3^l-1}{2}\varepsilon_0 > z. \qquad (12)$$

It is possibile, because $\varepsilon_0 > 0$ and $\varepsilon_1 > 0$ are sufficiently small numbers and hence $1-\varepsilon_1-\frac{3^l-1}{2}\varepsilon_0$ is sufficiently close to $1$ .

Now let $z \in (0,1)$ be an arbitrary number. Then we choose $y(t_0)=1-\varepsilon_1$ and $y(t_0+1)=1-\varepsilon_1-\varepsilon_2$ as it is described above and such that (12) is valid. Then the monotonous decreasing sequence $\{y(t)\}_{t=t_0}^{+\infty}$ satisfies (1), $1 > y(t) > y(t+1) > 0$ for $t \geq t_0$ , $t \in \mathrm{Z}$ and $y(t) \in (z,1)$ for $t = t_0$ , $t_0+1$ , ..., $t_0+l$ and $\lim y(t)=0$ for $t \to +\infty$ . Then there exists an positive integer $l_1 > l$ such that

$$\mathrm{y}(\mathrm{t}_0+\mathrm{l}_1-1) > \mathrm{z},\ \mathrm{y}(\mathrm{t}_0+\mathrm{l}_1) < \mathrm{z}. \qquad (13)$$

Now let us give the following observation. Let $\{y(t)\}$ and $\{z(t)\}$ are two solutions of (1), i.e. they satisfy (5). Then from (5)

$$y(t+1)=(2-k)y(t)+(k-1)y(t-1)-f(y(t))\ ;$$
$$z(t+1)=(2-k)z(t)+(k-1)z(t-1)-f(z(t))$$

and we substract the last equalities:

$$y(t+1)-z(t+1)=(2-k)(y(t)-z(t))-(f(y(t))-f(z(t)))+(k-1)(y(t-1)-z(t-1)). \qquad (14)$$

From (14) and C1 it follows that

$$\textit{if} \quad y(t-1)-z(t-1) \geq 0 \ \textit{and} \ y(t)-z(t) \geq 0 \ \textit{then} \ y(t+1)-z(t+1) \geq 0. \qquad (15)$$

Also if at least one of the first or the second inequalities of (15) is strong, then the third inequality of (15) is strong, i.e. $y(t+1) > z(t+1)$ . In particular if $y(t_0) \geq z(t_0)$ and $y(t_0+1) \geq z(t_0+1)$ and at least one of these two inequalities is strong, then $y(t) > z(t)$ for $t \geq t_0+2$ , $t \in \mathrm{Z}$. (In particular, putting $z(t) \equiv 0, \forall t \in Z$ , we obtain that $y(t) > 0, t \geq t_0$ ).

Let now as above $\varepsilon_2 > 0$ be small number. We define the function $\varepsilon_1(\varepsilon_2)$ so that

$$\mathrm{f}(1-\varepsilon_1(\varepsilon_2)-\varepsilon_2)=(\mathrm{k}-1)\ \varepsilon_2 \qquad (16)$$

and $\varepsilon_1(\varepsilon_2)$ is the smallest positive root of (16). Indeed, from C3 $f(1-\varepsilon_2) < (k-1)\varepsilon_2$ and from C2 such positive root exists $\forall \varepsilon_2 \in (0, \varepsilon_2^0]$ for some sufficiently small $\varepsilon_2^0 > 0$ .

Thus $\varepsilon_1(\varepsilon_2)$ is a continuous function for $\varepsilon_2 \in (0, \varepsilon_2^0]$, because from C3 $f$ is strictly monotonous decreasing function in a small left neibourhood of $1$. More precize from (16) $\varepsilon_1(\varepsilon_2)=1-\varepsilon_2-f^{-1}((k-1)\varepsilon_2) > 0$ , where $f^{-1}((k-1)\varepsilon_2)$ is the value of small left neighbourhood of $1$ and hence this function $\varepsilon_1(\varepsilon_2)$ is continuous function for $\varepsilon_2 \in (0, \varepsilon_2^0]$ and also $\varepsilon_1(\varepsilon_2) > 0$ is sufficiently small, when $\varepsilon_2 > 0$ is sufficiently small. For such choosen $\varepsilon_1(\varepsilon_2)$ , if for the solution $\{y(t)\}$ of (1) we choose as above

$$\mathrm{y}(\mathrm{t}_0)=1-\varepsilon_1,\ \mathrm{y}(\mathrm{t}_0+1)=1-\varepsilon_1-\varepsilon_2, \qquad (17)$$

then from (9) $y(t_0+2)=y(t_0+1)$ and $y(t) > y(t+1)$ for $t \geq t_0+2$ , $t \in \mathrm{Z}$ .

Let now define the solution $\{y_1(t)\}$ of (1) such that

$$y_1(t_0)=1-\varepsilon_1(\varepsilon_2^{(1)})\ ,\ \ y_1(t_0+1)=1-\varepsilon_1(\varepsilon_2^{(1)})-\varepsilon_2^{(1)}$$

and let $\varepsilon_2^{(1)} > 0$ be a number so small that inequalities (13) hold, where $z \in (0,1)$ is an arbitrary fixed real number and $l_1 > l$ , $l \in \mathrm{N}$ is an arbitrary fixed (sufficiently large) positive integer. Now we choose $\varepsilon_2^{(2)} \in \left(\varepsilon_2^{(1)}, \varepsilon_2^0\right)$ such that

$$\varepsilon_1\left(\varepsilon_2^{(2)}\right) > \varepsilon_1\left(\varepsilon_2^{(1)}\right) + \varepsilon_2^{(1)}. \tag{18}$$

It is possible, because the right hand-side of (18) is sufficiently small positive number for sufficiently small $\varepsilon_2^{(1)} > 0$ and $\varepsilon_1(\varepsilon_2)$ is positive for $\varepsilon_2 > 0$. Now we define an other solution $\{y_2(t)\}$ of (1) such that $y_2(t_0) = 1 - \varepsilon_1\left(\varepsilon_2^{(2)}\right)$ , $y_2(t_0+1) = 1 - \varepsilon_1\left(\varepsilon_2^{(2)}\right) - \varepsilon_2^{(2)}$ . Then from (18) $y_2(t_0) < y_1(t_0+1)$ and from (13), which are valid for $\{y_1(t)\}$ and (15) we obtain:

$$\mathrm{y}_2(\mathrm{t}_0 + \mathrm{l}_1 - 1) < \mathrm{y}_1(\mathrm{t}_0 + \mathrm{l}_1) < \mathrm{z} < \mathrm{y}_1(\mathrm{t}_0 + \mathrm{l}_1 - 1). \tag{19}$$

Let now $\{y(t)\}$ be an arbitrary solution of (1), for which (17) holds for $\varepsilon_1 = \varepsilon_1(\varepsilon_2)$ . Since $\varepsilon_1(\varepsilon_2)$ is a continuous function for $\varepsilon_2 \in \left(0, \varepsilon_2^0\right]$ , then $y(t_0 + l_1 - 1)$ is a continuous function with respect $y(t_0)$ and $y(t_0+1)$ and hence $y(t_0 + l_1 - 1)$ can be considered as a continuous function with respect $\varepsilon_2 \in \left(0, \varepsilon_2^0\right]$ , where $l_1$ is a fixed positive integer. Then $y_1(t)$ is obtained from $y(t)$ for $\varepsilon_2 = \varepsilon_2^{(1)}$ and $y_2(t)$ is obtained from $y(t)$ for $\varepsilon_2 = \varepsilon_2^{(2)}$ . From (19) there exists $\varepsilon_2^{(0)} \in \left(\varepsilon_2^{(1)}, \varepsilon_2^{(2)}\right)$ , such that if we define the solution $\{y_3(t)\}$ of (1) such that (17) is valid for $\varepsilon_1 = \varepsilon_1\left(\varepsilon_2^{(0)}\right)$ , then

$$y_3(t_0 + l_1 - 1) = z, y_3(t_0+2) = y_3(t_0+1) \ and \ y_3(t) > y_3(t+1) \quad for \quad t \geq t_0 + 2, t \in \mathrm{Z}. \tag{20}$$

Since $l_1 > l$ , where $l \in \mathrm{N}$ is an arbitrary fixed, sufficiently large positive integer, then from (20) the solution $\{y_3(t + l_1 - l - 1)\}_{t=t_0}^{+\infty}$ of (1) satisfies all conditions of Lemma 1 for $z_0 = y_3(t_0 + l_1 - l - 1)$ and we can choose $l_1 >> l$ . Lemma 1 is proved.

**HETEROCLINIC SOLUTIONS OF (1)**

If $y(t)$ is a solution of (1), then

$$\mathrm{y}(\mathrm{t}) - \mathrm{y}(\mathrm{t}-1) = -\frac{1}{\mathrm{k}-1}\left(\mathrm{y}(\mathrm{t}+1) - \mathrm{y}(\mathrm{t})\right) - \frac{1}{\mathrm{k}-1}\mathrm{f}\left(\mathrm{y}(\mathrm{t})\right) \tag{21}$$

and $y(t-1) = \frac{k-2}{k-1} y(t) + \frac{1}{k-1} y(t+1) + \frac{1}{k-1} f(y(t))$ , $k \in (1,2)$ . If $z(t)$ is another solution of (1), then we obtain:

$$\begin{aligned} \mathrm{y}(\mathrm{t}-1) - \mathrm{z}(\mathrm{t}-1) = -\frac{1}{\mathrm{k}-1}\Big[(2-\mathrm{k})\ \left(\mathrm{y}(\mathrm{t}) - \mathrm{z}(\mathrm{t})\right) - \left(\mathrm{f}\left(\mathrm{y}(\mathrm{t})\right) - \mathrm{f}\left(\mathrm{z}(\mathrm{t})\right)\right)\Big] \\ + \frac{1}{\mathrm{k}-1}\left(\mathrm{y}(\mathrm{t}+1) - \mathrm{z}(\mathrm{t}+1)\right) \ . \end{aligned} \tag{22}$$

From C1 the sign of $(2-k)(y(t) - z(t)) - (f(y(t)) - f(z(t)))$ coinsides with the sign of $y(t) - z(t)$ for $k \in (1,2)$ and we conclude that: If $y(t+1) - z(t+1) \geq 0$ ( $\leq 0$ ) and $y(t) - z(t) \leq 0$ ( $\geq 0$ ), where at least one of these inequalities is strong, then $y(t-1) - z(t-1) > 0$ ( $< 0$ ). In particular if $\{y(t)\}$ is a solution of (1), then $\{y(t+1)\}$ is also a solution of (1) and for $z(t) = y(t+1)$ , we can do the previous conclusion. If we have one of these situations, it follows that the sign of $y(t) - z(t)$ (or $y(t) - y(t+1)$ ) changes (oscilates) when $t \to -\infty$ , i.e. for $t = t_0$ , $t_0 - 1$ , $t_0 - 2$ , ... .And in each of these cases from (22) it follows that

$$|y(t-1) - z(t-1)| \geq \frac{1}{k-1}|y(t+1) - z(t+1)| \tag{23}$$

or

$$|y(t-1) - y(t)| \geq \frac{1}{k-1}|y(t+1) - y(t+2)|. \tag{24}$$

From (24) it follows that if $\{y(t)\}$ is a heteroclinic solution, then for any $t \in Z$ sufficiently small negative integer, it is not possibile that $y(t+1) - y(t+2) \geq 0$ and $y(t) - y(t+1) \leq 0$ , where at least one of these inequalities is strong and the converse case ( $y(t+1) - y(t+2) \leq 0$ and $y(t) - y(t+1) \geq 0$, where at least one of these inequalities is strong) is not possible, because from (24) since $\frac{1}{k-1} > 1$ for $k \in (1,2)$, it follows that $\lim|y(t+1) - y(t)| = \infty$ for $t \to -\infty$ and since the signs of $y(t-1) - y(t)$ change alternatively when $t \to -\infty$ , then $\lim\limits_{t \to -\infty} y(t)$ does not exists. Thus if $\{y(t)\}$ is a heteroclinic solution of (1), then $\{y(t)\}$ has to be monotonous. But if $y(t+1) > y(t)$ for some $t \in Z$ , then from (21) $y(t-1) - y(t) > 0$ , i.e. $y(t-1) > y(t) < y(t+1)$ and hence the sequence $\{y(t)\}$ cannot be monotonous $\forall t \in Z$ .

By analogy, if $y(t+1) = y(t) \in (0,1)$ for some $t \in Z$ , then from (21) $y(t-1) - y(t) > 0$ and we obtain

$$y(t-2) - y(t-1) = -\frac{1}{k-1}[(2-k)(y(t-1) - y(t)) - (f(y(t-1)) - f(y(t)))]$$
$$+\frac{1}{k-1}(y(t) - y(t+1)).$$

Since $\frac{1}{k-1}(y(t) - y(t+1)) = 0$ , then from C1,

$$y(t-2) - y(t-1)$$
$$= -\frac{1}{k-1}[(2-k)(y(t-1) - y(t)) - (f(y(t-1)) - f(y(t)))] < 0,$$

i.e. $y(t-2) < y(t-1) > y(t)$ and hence again $\{y(t)\}$ cannot be monotonous $\forall t \in Z$ . Thus if $\{y(t)\}$ is heteroclinic solution of (1) then

$$y(t) > y(t+1) \ \forall t \in Z \tag{25}$$

and $\lim\limits_{t \to \infty} y(t) = 0$ . We prove now that

$$y(t) \in (0,\ 1) \ \ \forall t \in Z. \tag{26}$$

Suppose the contrary, i.e. that $y(t_0) \in (0,1)$ , but $y(t_0 - 1) \geq 1$ . Then putting $z(t) \equiv 1$, $t \in Z$ we obtain that $y(t_0 - 2) < 1$ , $y(t_0 - 3) > 1$ , ..., i.e. $y(t_0) < y(t_0 - 1) > y(t_0 - 2)$, which contradics with (25). Thus we prove that (26) holds. If we take limit in (21) for $t \to -\infty$, then we obtain that $f(l_-) = 0$, where $l_- = \lim\limits_{t \to -\infty} y(t)$ and from (26) $l_- \in (0,1]$. Since $f(u) > 0$ for $u \in (0,1)$ , we conclude that the only possibility is $l_- = 1$ . Thus any heteroclinic solution of (1) satisfies conditions (25) and (26) and $\lim\limits_{t \to \infty} y(t) = 0$ , $\lim\limits_{t \to -\infty} y(t) = 1$ .

**Lemma 2.** *Let the function* $f$ *satisfy the assumptions C1, C2 and C3. Then for any* $y_1 \in (0,1)$, *there exists at most one heteroclinic solution* $\{y(t)\}$ *of (1) with the property* $y(1) = y_1$ .

**Proof:** Let $\{y(t)\}$ and $\{z(t)\}$ are two heteroclinic solutions of (1) for which $y(1) = z(1) = y_1$ . If $y(0) = z(0)$ then obviously $y(t) \equiv z(t)$ $\forall t \in \mathrm{Z}$ . Let us suppose for example that $y(0) > z(0)$ . Since $y(1) = z(1)$ , then from (22) $y(-1) < z(-1)$ , $y(-2) > z(-2)$ , ... and $|y(t-2) - z(t-2)| \geq \frac{1}{k-1}|y(t) - z(t)|$ $\forall t \in \mathrm{Z}$ . Since $\frac{1}{k-1} > 1$ for $k \in (1,2)$ , then it follows that $\lim_{t \to -\infty} |y(t) - z(t)| = \infty$ and then both $y(t)$ and $z(t)$ cannot be heteroclinic solutions of (1). Lemma 2 is proved.

Our aim now is to prove the existence of heteroclinic solution $\{y(t)\}$ of (1) for which $y(1) = y_1$ . Our main result is

**Theorem 3.** *Let the function* $f$ *satisfy the assumptions C1, C2 and C3. Then for any real number* $y_1 \in (0,1)$ *, there exists an unique heteroclinic solution of (1)* $\{y(t)\}$ *, satisfiyng the conditions:* $y(1) = y_1$ *,* $y(t) > y(t+1)$ *,* $\forall t \in \mathrm{Z}$ *,* $y(t) \in (0,1)$ *,* $t \in \mathrm{Z}$ *and* $\lim_{t \to +\infty} y(t) = 0$ *and* $\lim_{t \to -\infty} y(t) = 1$ *.*

**Proof:** Let $\{y(t)\}$ be a solution of (1) for which $y(1) = y_1 \in (0,1)$. Let $y(0) = y_0$ . If we denote $y(2) = y_2$ , then from (21) $y_2 - y_1 = -(k-1)(y_1 - y_0) - f(y_1) = (k-1)(y_0 - y_1) - f(y_1) < 0$ iff

$$\mathrm{y}_1 < \mathrm{y}_0 < \mathrm{y}_1 + \frac{\mathrm{f}(\mathrm{y}_1)}{\mathrm{k}-1} , \tag{27}$$

i.e. (27) holds if and only if $y_2 < y_1 < y_0$ . From (22) it is easy to obtain that

$$\begin{aligned} &\text{If } \ \mathrm{y}(\mathrm{t}-1) - \mathrm{z}(\mathrm{t}-1) \geq 0 \ (>0) \ \text{ and } \ \mathrm{y}(\mathrm{t}) - \mathrm{z}(\mathrm{t}) \geq 0 (>0), \\ &\text{then } \ \mathrm{y}(\mathrm{t}+1) - \mathrm{z}(\mathrm{t}+1) \geq 0 \ (>0) \ . \end{aligned} \tag{28}$$

So (27) is nessesary and sufficient condition for $y(t) > y(t+1), t = 0,1,...$ . Thus we obtained that $\{y(t)\}$ is monotonously decreasing solution of (1) for $t \geq 0$ if and only if (27) holds. Further for any fixed real number $y_0$ satisfying (27), we can obtain the solution $\{y(t)\}$ of (1) for $\forall t \in \mathrm{Z}$ . Our aim is to prove that there exists $y_0$ satisfying (27) such that the obtained solution $y(t) = y(t, y_1, y_0)$ of (1) satisfies the conditions (25) and (26). Let now for $\forall n \in \mathrm{N}$ with $\{y_n(t)\}$ we denote the solution of (1), satisfying Lemma 1, i.e. $y_n(1) = y_1$ and $y_n(t) > y_n(t+1)$ $\forall t \geq -n$ , $t \in \mathrm{Z}$, i.e. $y_n(t) \equiv y(t, y_1, y_n(0))$ and obviously $y_n(0)$ satisfies (27). Then the obtained sequence $\{y_n(0)\}_{n=1}^{\infty}$ is bounded and one can choose a convergent subsequence $\{y_{n_k}(0)\}_{k=1}^{\infty}$ for which $n_k \to \infty$ when $k \to \infty$ and $\lim_{k \to \infty} y_{n_k}(0) = \overline{y}_0 \in \left[y_1, y_1 + \frac{f(y_1)}{k-1}\right]$ , see (27). That means the solutions of (1) $\{y(t, y_1, y_{n_k}(0))\}_{k=1}^{\infty}$ have the properties $y(1, y_1, y_{n_k}(0)) = y_1$ , $y(0, y_1, y_{n_k}(0)) = y_{n_k}(0)$ and

$$\mathrm{y}\left(\mathrm{t},\ \mathrm{y}_1,\ \mathrm{y}_{\mathrm{n_k}}(0)\right) > \mathrm{y}\left(\mathrm{t}+1,\ \mathrm{y}_1,\ \mathrm{y}_{\mathrm{n_k}}(0)\right) \ \forall \mathrm{t} \geq -\mathrm{n_k},\ \mathrm{t} \in \mathrm{Z}. \tag{29}$$

We prove that $y(t) = y(t, y_1, \overline{y}_0)$ is the sought heteroclinic solution. We assume at first that there exists $s_0 \in \mathrm{N}$ such that $y(-s_0) > y(-s_0+1)$ , but $y(-s_0) > y(-s_0-1)$ , i.e.

$y\left(-s_0, y_1, \overline{y}_0\right) > y\left(-s_0+1, y_1, \overline{y}_0\right)$ and $y\left(-s_0, y_1, \overline{y}_0\right) > y\left(-s_0-1, y_1, \overline{y}_0\right)$. Since $s_0 \in \mathrm{N}$ is a fixed number, then $y\left(-s_0, y_1, y_0\right)$, $y\left(-s_0+1, y_1, y_0\right)$, $y\left(-s_0-1, y_1, y_0\right)$ are continuous functions with respect to $y_0$. Hence for sufficiently large numbers $n_k \in \mathrm{N}$,

$$\begin{aligned} &\mathrm{y}\left(-\mathrm{s}_0,\ \mathrm{y}_1,\ \mathrm{y}_{\mathrm{n_k}}(0)\right) > \mathrm{y}\left(-\mathrm{s}_0+1,\ \mathrm{y}_1,\ \mathrm{y}_{\mathrm{n_k}}(0)\right) \quad \text{and} \\ &\mathrm{y}\left(-\mathrm{s}_0,\ \mathrm{y}_1,\ \mathrm{y}_{\mathrm{n_k}}(0)\right) > \mathrm{y}\left(-\mathrm{s}_0-1,\ \mathrm{y}_1,\ \mathrm{y}_{\mathrm{n_k}}(0)\right). \end{aligned} \tag{30}$$

But (30) contradics to (29) for sufficiently large numbers $k$, such that $n_k >> s_0$. Hence the above assumption is not true. This fact, the fact that $\overline{y}_0 \in \left[y_1, y_1 + \frac{f(y_1)}{k-1}\right]$ and (28) show that the solution of (1) $y(t) = y\left(t, y_1, \overline{y}_0\right)$ thus defined is a monotonously nonincreasing sequence, i.e.

$$y(t) \geq y(t+1), \forall t \in \mathrm{Z}. \tag{31}$$

We assume now that for some $t_0 \in \mathrm{Z}$, $y(t_0) = y(t_0+1)$. Then from (21) we obtain $y(t_0-1) - y(t_0) = \frac{1}{k-1} f\left(y(t_0)\right) > 0$ and from (22) and C1, $y(t_0-2) - y(t_0-1) = -\frac{1}{k-1}\left[(2-k)\left(y(t_0-1)-y(t_0)\right) - \left(f\left(y(t_0-1)\right) - f\left(y(t_0)\right)\right)\right] < 0$, (because the last expression can be equal to $0$ iff $y(t_0-2) = y(t_0-1) = y(t_0) = y(t_0+1)$, i.e. $y(t) \equiv 0$ or $y(t) \equiv 1$ $\forall t \in \mathrm{Z}$, which is impossible). Thus we obtain that $y(t_0-2) < y(t_0-1) > y(t_0)$, which contradicts to (31). Thus (31) be in force, must all inequalities in (31) to be strong, i.e. for the our solution $y(t) = y\left(t, y_1, \overline{y}_0\right)$, (25) holds, i.e. $y(t) > y(t+1)$, $\forall t \in \mathrm{Z}$. Also if $y_1 \in (0,1)$, then $y_0 > y_1 > 0$, i.e. $y(0) > y(1) > 0$ and from the considerations in the previous Section it follows that $y(t) > 0$, $t \in \mathrm{Z}$. But we proved that if $\{y(t)\}$ satisfies (25), then (26) also holds. Thus we prove that the solution $y(t) = y\left(t, y_1, \overline{y}_0\right)$ of (1) satisfies (25) and (26) and as we show above,

$$\lim_{t\to+\infty} y(t) = 0 \quad \text{and} \quad \lim_{t\to-\infty} y(t) = 1, \tag{32}$$

i.e. $y(t)$ is the sought heteroclinic solution.

**Corollary 4.** *Let the function $f$ satisfy the assumptions C1, C2 and C3. Then for any real number $y_0 \in (0,1)$ and arbitrary $t_0 \in \mathrm{Z}$, there exists an unique heteroclinic solution of* (1) $\{y(t)\}$ *satisfiyng the conditions: $y(t_0) = y_0$, $y(t) > y(t+1)$, $\forall t \in \mathrm{Z}$, $y(t) \in (0,1)$, $\forall t \in \mathrm{Z}$ and $\lim_{t\to+\infty} y(t) = 0$ and $\lim_{t\to-\infty} y(t) = 1$.*

**Proof:** Let $\{y(t)\}$ be the heteroclinic solution of (1), for which $y(1) = y_0$ and $\{y(t)\}$ satisfies the other conditions of Theorem 3 - (25), (26) and (32). Then $\{y(t+1-t_0)\}$ is also a heteroclinic solution of (1), which also satisfies (25), (26), (32), and for $t = t_0$, $y(t+1-t_0) = y(1) = y_0$. The proof is completed.

**Example.** We give a class of functions, satisfying the assumptions C1, C2 and C3. Let $f(x) = c(2-k)(k-1)x^n(1-x)^n$, $x \in [0,1]$, where *c* and *n* are constant numbers such that $c \in (0,1]$, $n \geq 1$. Obviously *f* satisfies the assumptions C2 and C3. In order to prove that *f* satisfies the assumption C1, we have $\left(x^n(1-x)^n\right)' = n\left(x(1-x)\right)^{n-1}(1-2x)$ and since $x(1-x) \in [0,1/4]$ and $|1-2x| \leq 1$ for $x \in [0,1]$, we deduce that $\left|\left(x^n(1-x)^n\right)'\right| \leq n(1/4)^{n-1} = \dfrac{n}{4^{n-1}} \leq 1$ for $x \in [0,1]$, because it is easy to prove that $4^{n-1} \geq n$ for $n \in [1,\infty)$. Hence

$|f'(x)| \le (2-k)(k-1) < 2-k$ for $x \in [0,1]$, since $k \in (1,2)$. Thus we show that the assumption C1 is satisfied.

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## ХЕТЕРОКЛИНИЧНИ РЕШЕНИЯ НА ЕДНО ДИФЕРЕНЧНО УРАВНЕНИЕ ОТ ВТОРИ РЕД

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***Резюме:*** *В предлаганата статия се изследва проблемът за съществуване на хетероклинични решения за полулинейно диференчно уравнение от втори ред, свързано с уравнението на Фишер - Колмогоров $\Delta^2 y(t-1)+k\Delta y(t-1)+f(y(t))=0$ за $k\in(1,2)$. Аналогично уравнение се разглежда в* [5] *и тази работа е продължение на разглежданията в* [5]. *Доказателствата на представените резултати са базирани на съображения за монотонност и непрекъснатост.*

***Ключови думи:*** *Хетероклинични решения, Задача с начални условия, Диференчни уравнения*

ISSN 1314-3077
9 771314 307000