VARIOUS ASPECTS OF PI-THEORY

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Abstract: *If we interpret Algebra on a map and its historical development starting from the bottom as roads in it named (for example* ***A****(algebraic structures),* ***F****(field theory),* ***C****(commutative algebra),…,* ***AS****(associative rings and algebras),* ***G****(group theory) etc., these roads would have had studyalmost only vertical extensions going more and more into details of a specific topic. This survey gives an example of a possible horizontal algebraic route starting with involutions in matrix algebras and giving another approach to complex numbers making at the same time some small steps in vertical direction as well.*

Morover some subalgebras of the $n\times n$ matrix algebra over the Grassmann algebra are investigated and low degree identities for these algebras are discussed. A graphical illustration of the considered algebras for concrete n is made discussing the possible use of the system GeoGebra for it. Some applications of the approach are discussed as well presenting a new aspect of the classical PI-theory.

Key words**:** *Involution, symmetric and skew symmetric variables, φ-polynomial identities, Grassmann algebra, matrix algebras with Grassmann entries, varieties of algebras, standard polynomial, GeoGebra.*

1. INTRODUCTION

My long experience in teaching students at the university directed me to the opinion that it is not an easy answer what is more important – knowing facts or having ideas how facts could be combined in order the latter to be considered important and more understandable by wider range of people.

The paper gives an approach of how very specific ideas in PI-investigations made by the author could be interpreted in a way to give rise to classical algebraic objects thus illustrating in a better way specific study made. At the same time we investigate some varieties of algebras defined by low degree identities and give examples of subalgebras of the $n\times n$ matrix algebra over the Grassmann algebra belonging to the corresponding varieties. An illustration of the considered algebras for concrete n is made and some applications of the approach are discussed.

2. ALGEBRAIC ROUTES STARTING WITH PI - INVESTIGATIONS

EXPOSITION OF THE CASE $n=2$

It is well known [4, Corrolary 14.2] that in matrix algebras two involutions (second order antiautomorphisms) are important – the transpose one t and the symplectic one s. For second order matrix algebras (which we'll consider in this case) these are the following two:

The transpose involution $t(A)=t(\begin{pmatrix} a & b \\ c & d \end{pmatrix})=\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and the symplectic involution $s(A)=s(\begin{pmatrix} a & b \\ c & d \end{pmatrix})=\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Proposition 2.1 *The two involutions commute.*

Proof: This is straightforward as $t\circ s(A)=t(s(A))=t(\begin{pmatrix} d & -b \\ -c & a \end{pmatrix})=\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$ and $s\circ t(A)=s(t(A))=s(\begin{pmatrix} a & c \\ b & d \end{pmatrix})=\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$.

Proposition 2.2 *The mapping* $\varphi = t\circ s$ *is a homomorphism.*

Proof: Again a straightforward approach in proving gives that for any two matrices A and B we have $\varphi(AB)=t\circ s(AB)=t(s(AB))=t(s(B)s(A))=t(s(A))t(S(B))=\varphi(A)\varphi(B)$.

Proposition 2.3 *The mapping* φ *is a second order homomorphism.*

Proof: This could be shown either by the definition of φ i.e.

$$\varphi^2(A)=\varphi(\varphi(A))=\varphi(t\circ s(A))=\varphi(t(s(A)))=\varphi(\begin{pmatrix} d & -c \\ -b & a \end{pmatrix})=t\circ s(\begin{pmatrix} d & -c \\ -b & a \end{pmatrix})=\begin{pmatrix} a & b \\ c & d \end{pmatrix}=A$$

or using the properties of the involutions i.e.

$$\varphi^2(A)=t\circ s(t\circ s(A))=t\circ s\circ s\circ t(A))=t\circ t(A)=A.$$

Now we could find the symmetric $\{A_s\}$ and the skew symmetric $\{A_{ss}\}$ due to φ *elements of the matrix algebra* $M_2(K)$ over a field K of characteristic zero.

Proposition 2.4 *The symmetric due to* φ *elements of* $M_2(K)$ *are of type* $A_s=\begin{pmatrix} a_s & b_s \\ -b_s & a_s \end{pmatrix}$ *while the skew symmetric ones are of type* $A_{ss}=\begin{pmatrix} a_{ss} & b_{ss} \\ b_{ss} & -a_{ss} \end{pmatrix}$.

Proof: $\varphi(A)=\varphi(\begin{pmatrix} a & b \\ c & d \end{pmatrix})=\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ gives that $A_s=\begin{pmatrix} a_s & b_s \\ -b_s & a_s \end{pmatrix}$, while $\varphi(A)=\varphi(\begin{pmatrix} a & b \\ c & d \end{pmatrix})=\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}=\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$ gives that $A_{ss}=\begin{pmatrix} a_{ss} & b_{ss} \\ b_{ss} & -a_{ss} \end{pmatrix}$.

We point that in some sense we have symmetry and skew symmetry, respectively, due to the second diagonal.

We have the presentation $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}=A_s+A_{ss}=\begin{pmatrix} \frac{a+d}{2} & \frac{b-c}{2} \\ \frac{c-b}{2} & \frac{a+d}{2} \end{pmatrix}+\begin{pmatrix} \frac{a-d}{2} & \frac{b+c}{2} \\ \frac{b+c}{2} & \frac{d-a}{2} \end{pmatrix}$.

If φ is an involution it is important to consider separately the φ-symmetric elements $Y=\{Y_1,Y_2,...\}$ and the φ-skew symmetric elements $Z=\{Z_1,Z_2,...\}$ of an algebra as the first ones form a Jordan algebra due to the operation $Y_1\circ Y_2=Y_1Y_2+Y_2Y_1$ while the φ-skew symmetric ones form a Lie algebra due to the operation $[Z_1,Z_2]=Z_1Z_2-Z_2Z_1$.

Though in our case φ is not an antihomomorphism it is still interesting to investigate the properties of the elements in $\{A_s\}$ and $\{A_{ss}\}$ using now the traditional notations for them $Y=\{Y_1,Y_2,...\}$ and $Z=\{Z_1,Z_2,...\}$, respectively.

Proposition 2.5 The elements Y_1Y_2, Z_1Z_2 and $[Z_1,Z_2]$ are φ-symmetric elements.

Proposition 2.6 The elements YZ, ZY, $[Y,Z]$ and $Y \circ Z$ are φ-skew symmetric elements.

Proposition 2.7 The φ-symmetric elements of $M_2(K)$ commute. The inverse of a φ-symmetric element is a φ-symmetric element as well.

Thus we come in a different way to the classical problem that the set of the φ-symmetric elements of type $Y = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ is a field izomorphic to the field of the complex numbers.

This gives an example of a possible horisontal route on the map of Algebra starting with investigations concerning the properties of algebras with involutions and getting to school algebra.

We could continue our horisontal route considering transformations of the plane R^2 defined either by a φ- symmetric element or a φ-skew symmetric element.

Example 2.1 Let ψ be a transformation on the plane defined by $\begin{pmatrix} \psi(x) \\ \psi(y) \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ for given values $a, b \in K$.

a/ Find the images of chosen figures due to this transformation.

b/ Find the values of a, b for a specific, previously defined transformation.

Properties of the φ-symmetric elements of $M_2(G_1)$ when φ is an involution and G_1 is the odd part of the Grassmann algebra G are considered in [8]. Due to Proposition 9 in [8] the φ- symmetric elements *of* $M_2(G_1)$ anticommute.

SOME STEPS ON A VERTICAL ROUTE BASED ON THE CONSIDERATIONS MADE

In [5] we give the following

Definition 2.1. *Let $f = f(x_1, \ldots, x_m) \in K\langle x_1, ..., x_n \rangle$, the free associative algebra on n generators over K and φ be an involution. We say that f is a φ-identity in skew variables for the algebra A over K if $f = f(z_1, \ldots, z_m) = 0$ for all $z_1, ..., z_m \in \mathrm{A}^-$(the set of the φ-skew symmetric elements of A). Accordingly f is a φ-identity in symmetric variables for the algebra A over K if $f = f(y_1, \ldots, y_m) = 0$ for all $y_1, ..., y_m \in \mathrm{A}^+$ (the set of the φ-symmetric elements of A).*

The polynomial $f = f(x_1, \ldots, x_m)$ is a φ-identity for A if $f = f(u_1, \ldots, u_m) = 0$ for $u_1, ..., u_m \in \mathrm{A}^- \bigcup \mathrm{A}^+$ and there exist at least one i such that $u_i \in \mathrm{A}^-$ and at least one j such that $u_j \in \mathrm{A}^+$.

This definition could work as well when φ is a homomorphism and following [5] we find here the φ-identities of small degree in the algebra $M_2(F)$ equipped now with the homomorphism φ.

Proposition 2.7 leads to

Corollary 2.1 *The minimal degree of a φ-identity in symmetric variables for $M_2(K)$ is 2 and all φ-identities in symmetric variables are consequences of the identity $[Y_1, Y_2] = 0$.*

Proposition 2.8 *The minimal degree of a* φ*-identity in skew symmetric variables for* $M_2(K)$ *is 3 and all* φ*-identities in skew symmetric variables are consequences of the identities* $[Z_1^2, Z_2]=0$ *and* $[Z_1, Z_2]\circ Z=0$*.*

Proof: The validity of the proposition follows from the form of the matrices $Z^2=(a^2+b^2)E$ and $[Z_1, Z_2]=\begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix}$.

Corollary 2.2 *The minimal degree of a* φ*-identity for* $M_2(K)$ *is 3, namely* $[Y, Z^2]=0$*.*

We point that properties of the φ*-skew symmetric elements of* $M_2(G_1)$ are considered in [8, Corollary 4].

ON THE CASE $n=4$

We start with the definition of the symplectic involution in this case: For $A=(a_{ij})\in M_4(K)$ we have $s(A)=\begin{pmatrix} a_{33} & a_{43} & -a_{13} & -a_{23} \\ a_{34} & a_{44} & -a_{14} & -a_{24} \\ -a_{31} & -a_{41} & a_{11} & a_{21} \\ -a_{32} & -a_{42} & a_{12} & a_{22} \end{pmatrix}$.

Straight calculations show that in this case Propositions 2.1-2.3 are valid as well. Forming $\varphi(A)=t\circ s(A)=\begin{pmatrix} a_{33} & a_{34} & -a_{31} & -a_{32} \\ a_{43} & a_{44} & -a_{41} & -a_{42} \\ -a_{13} & -a_{14} & a_{11} & a_{12} \\ -a_{23} & -a_{24} & a_{21} & a_{22} \end{pmatrix}$ for $A\in M_4(K)$ we get easily the validity of

Proposition 2.9 *The set* $\{A_s\}$ *of the* φ*-symmetric elements of* $A\in M_4(K)$ *consists of matrices of type* $A_s=\begin{pmatrix} \alpha_s & a_s & b_s & c_s \\ d_s & \beta_s & e_s & f_s \\ -b_s & -c_s & \alpha_s & a_s \\ -e_s & -f_s & d_s & \beta_s \end{pmatrix}$*, while the set* $\{A_{ss}\}$ *of the* φ*-skew symmetric elements of* $A\in M_4(K)$ *consists of matrices of type* $A_{ss}=\begin{pmatrix} \alpha_{ss} & a_{ss} & b_{ss} & c_{ss} \\ d_{ss} & \beta_{ss} & e_{ss} & f_{ss} \\ b_{ss} & c_{ss} & -\alpha_{ss} & -a_{ss} \\ e_{ss} & f_{ss} & -d_{ss} & -\beta_{ss} \end{pmatrix}$ with elements from K.

We point that in this case we have the presentation

$$A = A_s + A_{ss}$$

$$= \begin{pmatrix} \frac{a_{11}+a_{33}}{2} & \frac{a_{12}+a_{34}}{2} & \frac{a_{13}-a_{31}}{2} & \frac{a_{14}-a_{32}}{2} \\ \frac{a_{21}+a_{43}}{2} & \frac{a_{22}+a_{44}}{2} & \frac{a_{23}-a_{41}}{2} & \frac{a_{24}-a_{42}}{2} \\ \frac{a_{31}-a_{13}}{2} & \frac{a_{32}-a_{14}}{2} & \frac{a_{11}+a_{33}}{2} & \frac{a_{12}+a_{34}}{2} \\ \frac{a_{41}-a_{23}}{2} & \frac{a_{42}-a_{24}}{2} & \frac{a_{21}+a_{43}}{2} & \frac{a_{22}+a_{44}}{2} \end{pmatrix} + \begin{pmatrix} \frac{a_{11}-a_{33}}{2} & \frac{a_{12}-a_{34}}{2} & \frac{a_{13}+a_{31}}{2} & \frac{a_{14}+a_{32}}{2} \\ \frac{a_{21}-a_{43}}{2} & \frac{a_{22}-a_{44}}{2} & \frac{a_{23}+a_{41}}{2} & \frac{a_{24}+a_{42}}{2} \\ \frac{a_{13}+a_{31}}{2} & \frac{a_{14}+a_{32}}{2} & \frac{a_{33}-a_{11}}{2} & \frac{a_{34}-a_{12}}{2} \\ \frac{a_{23}+a_{41}}{2} & \frac{a_{24}+a_{42}}{2} & \frac{a_{43}-a_{21}}{2} & \frac{a_{44}-a_{22}}{2} \end{pmatrix}.$$

As the set $\{A_s\}$ is closed under both the usual and the commutator multiplications we pay more attention to it.

We consider the subset of $\{A_s\}$ of those matrices for which $\alpha_s = \beta_s$ and $e_s = c_s$. For convenience we use for them the same notations $Y_1, Y_2, \ldots$.

We prove the following

Proposition 2.10 *In* $M_4(K)$ *we have the identity* $[[Y_1,Y_2]\circ[Y_3,Y_4],[Y_5,Y_6]\circ[Y_7,Y_8]] = 0$*.*

Proof: For the matrix $[Y_1,Y_2] = B = (b_{ij})$ we get

$$\begin{aligned} b_{11} &= a_1d_2 - a_2d_1 \\ b_{12} &= c_1(b_2 - f_2) - c_2(b_1 - f_1) \\ b_{13} &= -c_1(a_2 - d_2) + c_2(a_1 - d_1) \\ b_{14} &= -a_1(b_2 - f_2) + a_2(b_1 - f_1) \\ b_{21} &= -c_1(a_2 - d_2) + c_2(a_1 - d_1) = -b_{12} \\ b_{22} &= -a_1d_2 + a_2d_1 = -b_{11} \\ b_{23} &= d_1(b_2 - f_2) - d_2(b_1 - f_1) \\ b_{24} &= c_1(a_2 - d_2) - c_2(a_1 - d_1) = -b_{13} \end{aligned} ; \quad \begin{aligned} b_{31} &= c_1(a_2 - d_2) - c_2(a_1 - d_1) = -b_{13} \\ b_{32} &= a_1(b_2 - f_2) - a_2(b_1 - f_1) = -b_{14} \\ b_{33} &= a_1d_2 - a_2d_1 = b_{11} \\ b_{34} &= c_1(b_2 - f_2) - c_2(b_1 - f_1) = b_{12} \\ b_{41} &= -d_1(b_2 - f_2) + d_2(b_1 - f_1) = -b_{23} \\ b_{42} &= -c_1(a_2 - d_2) + c_2(a_1 - d_1) = b_{13} \\ b_{43} &= -c_1(b_2 - f_2) + c_2(b_1 - f_1) = -b_{12} \\ b_{44} &= -a_1d_2 + a_2d_1 = -b_{11} \end{aligned} .$$

For short a matrix of type B will be written as $B_i = \begin{pmatrix} x_i & y_i & z_i & t_i \\ -y_i & -x_i & u_i & -z_i \\ -z_i & -t_i & x_i & y_i \\ -u_i & z_i & -y_i & -x_i \end{pmatrix}$. Then we form $B_1 \circ B_2 = C = (c_{ij})$, for which

$$\begin{aligned} c_{11} &= 2x_1x_2 - 2y_1y_2 - 2z_1z_2 - t_1u_2 - t_2u_1 \\ c_{12} &= 0 \\ c_{13} &= 2x_1z_2 + 2x_2z_1 - y_1(t_2 - u_2) - y_2(t_1 - u_1) \\ c_{14} &= 0 \\ c_{21} &= 0 \\ c_{22} &= 2x_1x_2 - 2y_1y_2 - 2z_1z_2 - t_1u_2 - t_2u_1 = c_{11} \\ c_{23} &= 0 \\ c_{24} &= 2x_1z_2 + 2x_2z_1 - y_1(t_2 - u_2) - y_2(t_1 - u_1) = c_{13} \end{aligned} ;$$

$$c_{31} = -2x_1z_2 - 2x_2z_1 + y_1(t_2 - u_2) + y_2(t_1 - u_1) = -c_{13}$$
$$c_{32} = 0$$
$$c_{33} = 2x_1x_2 - 2y_1y_2 - 2z_1z_2 - t_1u_2 - t_2u_1 = c_{11}$$
$$c_{34} = 0$$
$$c_{41} = 0$$
$$c_{42} = -2x_1z_2 - 2x_2z_1 + y_1(t_2 - u_2) + y_2(t_1 - u_1) = -c_{13}$$
$$c_{43} = 0$$
$$c_{44} = 2x_1x_2 - 2y_1y_2 - 2z_1z_2 - t_1u_2 - t_2u_1 = c_{11}$$

A matrix of type C will be written as $C_i = \begin{pmatrix} m_i & 0 & n_i & 0 \\ 0 & m_i & 0 & n_i \\ -n_i & 0 & m_i & 0 \\ 0 & -n_i & 0 & m_i \end{pmatrix}$. Now it is easy to realize that $[C_1, C_2] = 0$, i.e. we get the identity $[[Y_1, Y_2] \circ [Y_3, Y_4], [Y_5, Y_6] \circ [Y_7, Y_8]] = 0$.

From a classical result of S. Amitsur and J. Levitzki from 1950 it follows that in $M_4(F)$ the minimal identity is $S_8(x_1, ..., x_8) = 0$, where $S_8(x_1, ..., x_8)$ is the standard polynomial of degree 8. We could rely Proposition 2.10 to it writing the proved identity as $S_2(S_2(Y_1, Y_2) \circ S_2(Y_3, Y_4), S_2(Y_5, Y_6) \circ S_2(Y_7, Y_8)) = 0$, which is of the same degree 8.

It is difficult to find identities in φ-skew symmetric elements in this case different from the above standard identity. Following the approach used investigating symmetric elements, while considering the subset of $\{A_{ss}\}$ of those matrices for which $\alpha = \beta$ and $e = c$ we get that their commutator B_i still has 6 parameters, namely

$$B_i = \begin{pmatrix} x_i & y_i & z_i & t_i \\ -y_i & -x_i & u_i & v_i \\ -z_i & -t_i & x_i & y_i \\ -u_i & -v_i & -y_i & -x_i \end{pmatrix}.$$

The same is valid both for $[B_1, B_2] = (c_{ij}) = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & -c_{11} & c_{23} & -c_{13} \\ -c_{13} & -c_{14} & c_{11} & c_{12} \\ -c_{23} & c_{13} & c_{21} & -c_{11} \end{pmatrix}$ and $B_1 \circ B_2 = (d_{ij})$, where at least $d_{11}, d_{12}, d_{13}, d_{14}, d_{21}, d_{22}$ are different. The corresponding 6 matrix elements are different in the matrix $A_{ss1} \circ A_{ss2}$ as well.

We have analogous difficulties considering a 5-dimensional subset of $\{A_{ss}\}$, namely the matrices Z_i for which $\alpha_{ss} = \beta_{ss}, e_{ss} = c_{ss}, b_{ss} = f_{ss}$. For $[Z_1, Z_2] = (a_{ij})$ we get

$$\begin{array}{ll} a_{11}=a_1d_2-a_2d_1 & a_{31}=-a_{13} \\ a_{12}=0 & a_{32}=-a_{14} \\ a_{13}=2\alpha_1b_2-2\alpha_2b_1-c_1(a_2+d_2)+c_2(a_1+d_1) & a_{33}=a_{11} \\ a_{14}=2\alpha_1c_2-2\alpha_2c_1+2a_1b_2-2a_2b_1 & a_{34}=0 \\ a_{21}=0 & a_{41}=-a_{23} \\ a_{22}=-a_{11} & a_{42}=-a_{13} \\ a_{23}=-2\alpha_1c_2-2\alpha_2c_1+2d_1b_2-2d_2b_1 & a_{43}=0 \\ a_{24}=a_{13} & a_{44}=-a_{11} \end{array};\quad .$$

For short we write $A_i=\begin{pmatrix} x_i & 0 & y_i & z_i \\ 0 & -x_i & t_i & y_i \\ -y_i & -z_i & x_i & 0 \\ -t_i & -y_i & 0 & -x_i \end{pmatrix}$. As all the elements of both the matrices $[A_1,A_2]$ and $A_1\circ A_2$ are still nonzero we leave this case as not being a good example on the route of finding identities of small degree.

As an illustration of an identity with variables from $\{A_{ss}\}$ we use the following 4-dimensional subset of $\{A_{ss}\}$ namely the one of the matrices for which $\alpha_{ss}=\beta_{ss}, e_{ss}=c_{ss}$, $b_{ss}=f_{ss}$ and $a_{ss}=d_{ss}$. These are the matrices of type $Z=\begin{pmatrix} \alpha_{ss} & a_{ss} & b_{ss} & c_{ss} \\ a_{ss} & \alpha_{ss} & c_{ss} & b_{ss} \\ b_{ss} & c_{ss} & -\alpha_{ss} & -a_{ss} \\ c_{ss} & b_{ss} & -a_{ss} & -\alpha_{ss} \end{pmatrix}$.

We have $[Z_1,Z_2]=\begin{pmatrix} 0 & 0 & y & z \\ 0 & 0 & z & y \\ -y & -z & 0 & 0 \\ -z & -y & 0 & 0 \end{pmatrix}$.

As $[Z_1,Z_2][Z_3,Z_4]=\begin{pmatrix} -y_1y_2-z_1z_2 & -y_1z_2-y_2z_1 & 0 & 0 \\ -y_1z_2-y_2z_1 & -y_1y_2-z_1z_2 & 0 & 0 \\ 0 & 0 & -y_1y_2-z_1z_2 & -y_1z_2-y_2z_1 \\ 0 & 0 & -y_1z_2-y_2z_1 & -y_1y_2-z_1z_2 \end{pmatrix}$ we get the identity $[[Z_1,Z_2],[Z_3,Z_4]]=0$.

Thus we have shown that the elements of the considered 4-dimensional subset satisfy the above Lie identity of the same degree 4.

Still it is an open question to find an analogue of Corollary 2.2 i.e. a φ-identity of minimal degree for $M_4(K)$. Considering the 8-parameters both in a symmetric element Y and in a skew symmetric element Z we see that both the matrices $Y\circ Z$ and $[Y,Z]$ keep nonzero the same number of these parameters.

3. THE GRASSMANN IDENTITY $[X,Y,Z]=0$ AND MATRIX VARIETIES CONNECTED WITH IT

THE MATRIX VARIETY $\Re_1$ DEFINED BY THE IDENTITY $[X,Y,Z]U=0$

In several papers [6,7,9,10] the author investigated the PI-properties of some matrix algebras with Grassmann entries (elements from the Grassmann algebra). We recall the definition of the infinite dimensional Grassmann algebra E as

$$E=E(V)=K\langle e_1,e_2,\ldots \mid e_ie_j+e_je_i=0 \quad i,j=1,2,\ldots\rangle,$$

where the field K has characteristic zero.

The significance of considering the matrix algebra with Grassmann entries $M_n(E)$ is confirmed by the following statement as the trivial isomorphism $E\otimes M_n(K)\cong M_n(E)$ holds:

Proposition 3.1 [2, Corollary 8.2.4]: *For every PI-algebra R there exists a positive n such that $T(R)\supseteq T(M_n(E))$, i.e. R satisfies all polynomial identities of the $n\times n$ matrix algebra $M_n(E)$ with entries from the Grassmann algebra.*

Many of the PI-properties of E and $M_n(E)$ could be found in [1,3]. Here we formulate:

Proposition 3.2 [3, Corollary, p. 437]: *The T-ideal $Id(E)$ is generated by the identity $[x_1,x_2,x_3]=[x_1,x_2]x_3-x_3[x_1,x_2]=0$ (known as the Grassmann identity).*

Proposition 3.3 [1, Lemma 6.1]: *The algebra E satisfies the identity $S_n(x_1,\ldots,x_n)^k=0$ for all $n,k\geq 2$ and $S_n(x_1,\ldots,x_n)=\sum_{\sigma\in Sym(n)}(-1)^\sigma x_{\sigma(1)}\ldots x_{\sigma(n)}$ being the standard polynomial.*

Proposition 3.4 [1, Corollary 6.6]: *The algebra $M_n(E)$ does not satisfy the identity $S_m^n(X_1,\ldots,X_m)=0$ for any m.*

In this section we investigate some varieties of algebras defined by low degree identities and give examples of subalgebras of the $n\times n$ matrix algebra over the Grassmann algebra belonging to the corresponding varieties. An illustration of the considered algebras for concrete n is made and some applications of the approach are discussed.

We point that all commutative matrix algebras with Grassmann entries satisfy the Grassmann identity $[X,Y,Z]=0$. We give one example only – the X-figural subalgebra of $M_5(G)$, consisting of the matrices of type

$$\begin{pmatrix} a & 0 & 0 & 0 & a \\ 0 & b & 0 & b & 0 \\ 0 & 0 & c & 0 & 0 \\ 0 & b & 0 & b & 0 \\ a & 0 & 0 & 0 & a \end{pmatrix}$$ with a graphical way of presenting

$$\begin{matrix} \times & & & & \times \\ & * & & * & \\ & & \bullet & & \\ & * & & * & \\ \times & & & & \times \end{matrix}.$$

That is why we continue with only noncommutative matrix algebras.

The matrix variety $\Re_1$ defined by the identity $[X,Y,Z]U=0$ is considered in details in [10]. Here we give examples of algebras belonging to this variety:

Example 3.1 Let $\alpha_2,...,\alpha_n$ be fixed elements of the field K. We consider the n-th dimensional matrix algebra $A_0(E)$ of the matrices of type $\begin{pmatrix} x_1 & x_2 & ... & x_n \\ \alpha_2 x_1 & \alpha_2 x_2 & ... & \alpha_2 x_n \\ ... & ... & ... & ... \\ \alpha_n x_1 & \alpha_n x_2 & ... & \alpha_n x_n \end{pmatrix}$.

This algebra was investigated in [10] where the following theorem was proved, namely

Theorem 3.1 [10] *The algebra* $A_0(E)$ *belongs to the variety* $\Re_1$.

Example 3.2 Let $A_1(E)$ be the $(2n+1)\times(2n+1)$ matrix algebra of the matrices of type

$$\begin{pmatrix} a_1 & 0 & ... & ... & ... & 0 & a_{2n+1} \\ 0 & a_2 & 0 & ... & 0 & a_{2n} & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & ... & 0 & a_{n+1} & 0 & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & a_2 & 0 & ... & 0 & a_{2n} & 0 \\ a_1 & 0 & ... & ... & ... & 0 & a_{2n+1} \end{pmatrix} : a_j \in E,\ j=1,...,2n+1.$$

The algebra $A_1(E)$ is with basis

$$\begin{array}{lll} f_1 & = & e_{11}+e_{2n+1,1} \\ f_2 & = & e_{22}+e_{2n,2} \\ ... & ... & ... \\ f_n & = & e_{nn}+e_{n+2,n} \\ f_{n+1} & = & e_{n+1,n+1} \\ f_{n+2} & = & e_{n,n+2}+e_{n+2,n+2} \\ ... & ... & ... \\ f_{2n+1} & = & e_{1,2n+1}+e_{2n+1,2n+1} \end{array}.$$

This basis is a multiplicative one with properties $f_i f_j = f_j$ for eihter $i=j$ or $i+j=2n+2$ and $f_i f_j = 0$ for $i\neq j$ or $i+j\neq 2n+2$.

The algebra $A_1(E)$ satisfies the property that the sum of the entries in each row of the matrix $[X,Y,Z]$ for $X,Y,Z\in A_1(E)$ is zero. This could be shown in the following way:

Let denote by a_i the corresponding nonzero elements of the matrix X, by b_i the corresponding nonzero elements of the matrix Y and by c_i the corresponding nonzero elements of the matrix Z. Direct calculations show that in $[X,Y,Z]=(m_{ij})$ modulo the Grassmann identity we have

$\sum_{i=1}^{2n+1} m_{ki} = [[a_k,b_{2n+2-k}]+[a_{2n+2-k},b_k],c_k+c_{2n+2-k}]=0,\ k\neq n+1, 1\leq k\leq 2n+1$ and

$m_{n+1,n+1}=[a_{n+1},b_{n+1},c_{n+1}]=0$ as well.

Thus the matrix multiplication rule gives both the identity $[X,Y,Z]U=0$ and the validity of the following

Theorem 3.2 *The algebra* $A_1(E)$ *belongs to the variety* $\mathfrak{R}_1$.

The graphical way of describing the elements of the algebra $A_1(E)$ for $n=2$ gives

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Let $A_2(E)$ be the algebra of the matrices of type

$$\begin{pmatrix} a_1 & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & 0 & a_{2n+1} \\ 0 & a_2 & 0 & \ldots & \ldots & \ldots & 0 & a_{2n} & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & a_n & 0 & a_{n+2} & 0 & \ldots & 0 \\ a_1 & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & 0 & a_{2n+1}. \\ 0 & \ldots & 0 & a_n & 0 & a_{n+2} & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & a_2 & 0 & \ldots & \ldots & \ldots & 0 & a_{2n} & 0 \\ a_1 & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & 0 & a_{2n+1} \end{pmatrix} : a_j \in E,\ j \neq n+1,\ j=1,\ldots,2n+1.$$

Theorem 3.3 *The algebra* $A_2(E)$ *belongs to the variety* $\mathfrak{R}_1$.

Proof: The considerations are similar as above. Modulo the Grassmann identity we have

$$\sum_{i=1}^{2n+1} m_{ki} = [[a_k, b_{2n+2-k}] + [a_{2n+2-k}, b_k], c_k + c_{2n+2-k}] = 0,\ k \neq n+1, 1<k<2n+1,$$

$$\sum_{i=1}^{2n+1} m_{n+1,i} = \sum_{i=1}^{2n+1} m_{1i} = \sum_{i=1}^{2n+1} m_{2n+1,i}$$

and thus the algebra $A_2(E)$ satisfies the identity $[X,Y,Z]U=0$.

In this case the above graphical way gives the following picture:

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$\circ$ $\qquad$ $*$

$\times$ $\qquad$ $\div$

Different systems for illustration, like GeoGebra for example, could be used for presenting the considered algebras $A_1(E)$ and $A_2(E)$ for concrete n in more picturesque way.

Detailed results in this direction will be shown in another publication.

THE MATRIX VARIETY $\Re_2$ DEFINED BY THE IDENTITY $U[X,Y,Z]=0$

Example 3.3 Let $B_1(E)$ be the X-figural $(2n+1)\times(2n+1)$ matrix algebra of the matrices of type

$$\begin{pmatrix} a_1 & 0 & \dots & \dots & \dots & 0 & a_1 \\ 0 & a_2 & 0 & \dots & 0 & a_2 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & a_{n+1} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & a_{2n} & 0 & \dots & 0 & a_{2n} & 0 \\ a_{2n+1} & 0 & \dots & \dots & \dots & 0 & a_{2n+1} \end{pmatrix} : a_j \in E,\ j=1,\dots,2n+1.$$

Theorem 3.4 [10] *The algebra* $B_1(E)$ *belongs to the variety* $\Re_2$.

To the same variety belongs as well the algebra $B_2(E)$ of the matrices of the following type, namely

$$\begin{pmatrix} a_1 & 0 & \dots & \dots & \dots & \dots & \dots & 0 & a_1 \\ 0 & a_2 & 0 & \dots & \dots & \dots & 0 & a_2 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & a_n & 0 & a_n & 0 & \dots & 0 \\ a_{n+1} & 0 & \dots & \dots & \dots & \dots & \dots & 0 & a_{n+1}\ . \\ 0 & \dots & 0 & a_{n+2} & 0 & a_{n+2} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & a_{2n} & 0 & \dots & \dots & \dots & 0 & a_{2n} & 0 \\ a_{2n+1} & 0 & \dots & \dots & \dots & \dots & \dots & 0 & a_{2n+1} \end{pmatrix} : a_j \in E,\ j=1,\dots,2n+1.$$

The corresponding graphical representations of the algebras $B_1(E)$ and $B_2(E)$ for $n=2$ are

$$\begin{array}{ccccc} \times &&&& \times \\ & \circ && \circ & \\ && \bullet && \\ & * && * & \\ \div &&&& \div \end{array} \quad \text{and} \quad \begin{array}{ccccc} \times &&&& \times \\ & \circ && \circ & \\ \bullet &&&& \bullet \\ & * && * & \\ \div &&&& \div \end{array}$$

, respectively.

Now we consider the subalgebra $B_3(E)$ of $M_8(E)$ of the matrices of type

$$\begin{pmatrix} a & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & d & 0 & 0 & 0 & d \\ e & 0 & 0 & 0 & e & 0 & 0 & 0 \\ 0 & f & 0 & 0 & 0 & f & 0 & 0 \\ 0 & 0 & g & 0 & 0 & 0 & g & 0 \\ 0 & 0 & 0 & h & 0 & 0 & 0 & h \end{pmatrix}$$

and prove that it is a PI-algebra.

Theorem 3.5 *The algebra* $B_3(E)$ *belongs to the variety* $\Re_2$.

Proof: Let X,Y,Z be matrices from $B_3(E)$ with entries $a_i, b_i, ..., h_i$ for $i=1,2,3$, respectively. We form the diagonal entries of $[X,Y]=(a_{ij})$, namely

$$\begin{aligned} a_{11} &= [a_1,a_2]+a_1e_2-a_2e_1 \\ a_{22} &= [b_1,b_2]+b_1f_2-b_2f_1 \\ &\ \ ... \\ a_{88} &= [h_1,h_2]+h_1d_2-h_2d_1 \end{aligned}.$$

For the matrix $[X,Y,Z]=(b_{ij})$ we have modulo $[x,y,z]=0$ for $x,y,z\in E$ that

$$\begin{aligned} & b_{11}+b_{51} \\ &= [a_1e_2-a_2e_1,a_3]+([a_1,a_2]+a_1e_2-a_2e_1)e_3-a_3([e_1,e_2]+e_1a_2-e_2a_1) \\ &+[e_1a_2-e_2a_1,e_3]+([e_1,e_2]+e_1a_2-e_2a_1)a_3-e_3([a_1,a_2]+a_1e_2-a_2e_1) \\ &= [a_1e_2-a_2e_1,a_3]+(a_1e_2-a_2e_1)e_3-a_3(e_1a_2-e_2a_1) \\ &+[e_1a_2-e_2a_1,e_3]+(e_1a_2-e_2a_1)a_3-e_3(a_1e_2-a_2e_1) \\ &= [a_1e_2-a_2e_1,a_3]+[e_1a_2-e_2a_1,e_3]+[a_1e_2-a_2e_1,e_3]+[e_1a_2-e_2a_1,a_3] \\ &= [[a_1,e_2]+[e_1,a_2],a_3]+[[e_1,a_2]+[a_1,e_2],e_3]\equiv 0. \end{aligned}$$

Analogously we get that $b_{22}+b_{62}=b_{33}+b_{73}=b_{44}+b_{84}=0$, i.e. the sum of the entries in each column of $[X,Y,Z]$ is zero. The type of any matrix $U\in B_2(E)$ and the matrix multiplication rule give that $U[X,Y,Z]=0$.

This is easily seen by the following graphical representation of the matrices U and $[X,Y,Z]$, namely

$$\begin{matrix} \times & & & & \times & & & \\ & * & & & & * & & \\ & & \bullet & & & & \bullet & \\ & & & \circ & & & & \circ \\ \div & & & & \div & & & \\ & \oplus & & & & \oplus & & \\ & & \square & & & & \square & \\ & & & \square & & & & \square \end{matrix} \quad \text{and} \quad \begin{matrix} \times & & & & \times & & & \\ & * & & & & * & & \\ & & \bullet & & & & \bullet & \\ & & & \circ & & & & \circ \\ -\times & & & & -\times & & & \\ & -* & & & & -* & & \\ & & -\bullet & & & & -\bullet & \\ & & & -\circ & & & & -\circ \end{matrix}$$, respectively.

The analogue of $B_3(E)$ in the general case - the matrix algebra $B_{2,2n}(E)=(b_{ij})$, where $b_{ii}=b_{i,n+1}$ for $i=1,...,n$ and $b_{jj}=b_{j,n-j}$ for $j=n+1,...,2n$ satisfies the same identity $U[X,Y,Z]=0$.

THE VARIETY $\Re_3$ DEFINED BY THE IDENTITY $[X_1, Y_1, Z_1]U[X_2, Y_2, Z_2] = 0$

Let $C_1(E)$ be the $(4n+1)$-th dimensional matrix algebra of the matrices of type

$$\begin{pmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ a_{n1} & 0 & \dots & 0 \\ a_{n+1,1} & a_{n+1,2} & \dots & a_{n+1,2n+1} \\ a_{n+2,1} & 0 & \dots & 0 \\ a_{n+3,1} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ a_{2n+1,1} & 0 & \dots & 0 \end{pmatrix}$$

with a graphical representation for $n = 2$ the following

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one, namely $* \quad \oplus \quad \bullet \quad □ \quad \mp$.

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Theorem 3.6 [7] *The matrix algebra* $C_1(E)$ *belongs to the variety* $\Re_3$.

Proof: Let $X_1, Y_1, Z_1, U \in C_1(E)$. In $[X_1, Y_1, Z_1]U = (s_{ij})$ the only nonzero elements are s_{k1} for any k while in $[X_1, Y_1, Z_1] = (m_{ij})$ we have $m_{11} = m_{n+1,n+1} = 0$. Thus we get $[X_1, Y_1, Z_1]U[X_2, Y_2, Z_2] = 0$.

We point that if we change the places of the $(n+1)$-th row and of any of the other ones in any matrix from the algebra $C_1(E)$ we get representatives of $2n$ more classes of algebras all of which belong to the variety $\Re_3$.

Let $C_2(E)$ be the algebra of the matrices of type

$$\begin{pmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ a_{n1} & 0 & \dots & 0 \\ a_{n+1,1} & a_{n+1,2} & \dots & a_{n+1,2n+1} \\ 0 & 0 & \dots & a_{n+2,2n+1} \\ 0 & 0 & \dots & a_{n+3,2n+1} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{2n+1,2n+1} \end{pmatrix}$$

and its

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graphical representation for $n = 2$ the following one: $* \quad \oplus \quad \bullet \quad □ \quad \mp$.

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Analogously to the proof of Theorem 3.6 we get the validity of

Theorem 3.7 *The matrix algebra* $C_2(E)$ *belongs to the variety* $\mathfrak{R}_3$.

Again if we change the places of the $(n+1)$-th row and of any of the other ones for all the matrices of $C_2(E)$ we get representatives of $2n$ more classes of algebras all of which belong to $\mathfrak{R}_3$.

For simplicity the next considerations will be made for 5×5 matrices.

Let $C_3(E)$ be the algebra of the matrices of type $\begin{pmatrix} 0 & 0 & a_{13} & 0 & 0 \\ 0 & 0 & a_{23} & 0 & 0 \\ 0 & 0 & a_{33} & 0 & 0 \\ 0 & 0 & a_{43} & 0 & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix}$,

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Theorem 3.8 *The algebra* $C_3(E)$ belongs to the variety $\mathfrak{R}_3$.

Proof: Let $X_1, Y_1, Z_1, U \in C_3(E)$. For $[X_1, Y_1, Z_1] = (m_{ij})$ we have $m_{55} = 0$, while in $U[X_1, Y_1, Z_1] = (s_{ij})$ the only nonzero elements are $s_{51}, s_{52}, s_{53}, s_{54}$. Thus the matrix multiplication rule gives that $[X_1, Y_1, Z_1]U[X_2, Y_2, Z_2] = 0$.

We point that the identity $[X_1, Y_1, Z_1]U[X_2, Y_2, Z_2] = 0$ holds for the $(2n+1)\times(2n+1)$ analogue of the considered algebra $C_3(E)$ as well.

Let $C_4(E)$ be the algebra of the matrices of type $\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & 0 & a_{23} & 0 & 0 \\ 0 & 0 & a_{33} & 0 & 0 \\ 0 & 0 & a_{43} & 0 & 0 \\ 0 & 0 & a_{53} & 0 & 0 \end{pmatrix}$ with a

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Theorem 3.9 *The algebra* $C_4(E)$ belongs to the variety $\mathfrak{R}_3$.

Proof: Let $X_1, Y_1, Z_1, U \in C_4(E)$. For $[X_1, Y_1, Z_1] = (m_{ij})$ we have $m_{11} = 0$, while in $U[X_1, Y_1, Z_1] = (s_{ij})$ the only nonzero elements are $s_{12}, s_{13}, s_{14}, s_{15}$. Thus we get $[X_1, Y_1, Z_1]U[X_2, Y_2, Z_2] = 0$.

The same is valid for the $(2n+1)\times(2n+1)$ analogue of the considered algebra $C_4(E)$ as well.

THE VARIETY $\Re_4$ DEFINED BY THE IDENTITY $[X_1,Y_1,Z_1][X_2,Y_2,Z_2]U=0$

Let $D_1(E)$ be the algebra of the matrices of type $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ 0 & a_{42} & 0 & a_{44} & 0 \\ a_{51} & 0 & 0 & 0 & a_{55} \end{pmatrix}$. Its graphical reprsentation is $\begin{matrix} * & \oplus & \bullet & \square & \mp \\ & \circ & & \otimes & \\ \times & & & & \div \end{matrix}$.

Theorem 3.10 *The matrix algebra $D_1(E)$ belongs to the variety $\Re_4$.*

Proof: Let $X_1,Y_1,Z_1,U \in D_1(E)$. In $[X_1,Y_1,Z_1]=(m_{ij})$ we have $m_{i3}=0$ for all i, while in $[X_1,Y_1,Z_1]U=(s_{ij})$ the only nonzero elements are $s_{31},s_{32},s_{34},s_{35}$. Thus we get $[X_1,Y_1,Z_1][X_2,Y_2,Z_2]U=0$.

Let $D_2(E)$ be the algebra of the matrices of type $\begin{pmatrix} a_{11} & 0 & 0 & 0 & a_{15} \\ 0 & a_{22} & 0 & a_{24} & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ with a graphical reprsentation $\begin{matrix} \times & & & & \div \\ & \circ & & \otimes & \\ * & \oplus & \bullet & \square & \mp \end{matrix}$.

Theorem 3.11 *The matrix algebra $D_2(E)$ belongs to the variety $\Re_4$.*

Proof: The considerations are similar to the ones in the proof of Theorem 3.10.

Let $X_1,Y_1,Z_1,U \in D_2(E)$. In $[X_1,Y_1,Z_1]=(m_{ij})$ we have $m_{i3}=0$ for all i, while in $[X_1,Y_1,Z_1]U=(s_{ij})$ the only nonzero elements are $s_{31},s_{32},s_{34},s_{35}$. Thus we get $[X_1,Y_1,Z_1][X_2,Y_2,Z_2]U=0$.

We point that to the variety $\Re_4$ belong the $(2n+1)\times(2n+1)$ analogues of the algebras $D_1(E)$ and $D_2(E)$ as well.

THE VARIETY $\mathfrak{R}_5$ DEFINED BY THE IDENTITY $U[X_1,Y_1,Z_1][X_2,Y_2,Z_2]=0$

Let $T_1(E)$ be the algebra of the matrices of type $\begin{pmatrix} 0 & 0 & a_{13} & 0 & a_{15} \\ 0 & 0 & a_{23} & a_{24} & 0 \\ 0 & 0 & a_{33} & 0 & 0 \\ 0 & 0 & a_{43} & a_{44} & 0 \\ 0 & 0 & a_{53} & 0 & a_{55} \end{pmatrix}$ and

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$\oplus$	$\circ$	
$\bullet$		
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$\mp$		$\div$

be its graphical reprsentation.

Theorem 3.12 *The matrix algebra $T_1(E)$ belongs to the variety $\mathfrak{R}_5$.*

Proof: Let $X_i,Y_i,Z_i,U\in T_1(E)$ for $i=1,2$. In $[X_1,Y_1,Z_1][X_2,Y_2,Z_2]=(m_{ij})$ the only nonzero elements are m_{13} and m_{23}, while in U the first and the second column contain only zero elements.Thus we get $U[X_1,Y_1,Z_1][X_2,Y_2,Z_2]=0$.

Let $T_2(E)$ be the algebra of the matrices of type

$$\begin{pmatrix} a_{11} & 0 & a_{13} & 0 & 0 \\ 0 & a_{22} & a_{23} & 0 & 0 \\ 0 & 0 & a_{33} & 0 & 0 \\ 0 & a_{42} & a_{43} & 0 & 0 \\ a_{51} & 0 & a_{53} & 0 & 0 \end{pmatrix}$$

with a graphical representation

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	$\circ$	$\oplus$
		$\bullet$
	$\otimes$	□
$\div$		$\mp$

.

Theorem 3.13 *The matrix algebra $T_2(E)$ belongs to the variety $\mathfrak{R}_5$.*

Proof: Let $X_i,Y_i,Z_i,U\in T_1(E)$ for $i=1,2$. In $[X_1,Y_1,Z_1][X_2,Y_2,Z_2]=(m_{ij})$ the only nonzero elements are m_{43} and m_{53}, while in U the fourth and the fifth column contain only zero elements.Thus we get $U[X_1,Y_1,Z_1][X_2,Y_2,Z_2]=0$.

The $(2n+1)\times(2n+1)$ analogues of the algebras $T_1(E)$ and $T_2(E)$ belong to the same variety $\mathfrak{R}_5$.

Corrolary 3.1 *In all the considered varieties the triple commutator is nilpotent.*

For the algebras from the varieties $\mathfrak{R}_1$ and $\mathfrak{R}_2$ its index of nilpotency is 2, while for the last three varieties it is 3. The proof is straightforward.

Remark 3.1 *We point two more classes of algebras, presented by the following two types of 5×5 matrices (partial cases of the corresponding $(2n+1)\times(2n+1)$ analogues), namely*

$$F_1 = \begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ 0 & 0 & 0 & a_{44} & 0 \\ 0 & 0 & 0 & 0 & a_{55} \end{pmatrix} \text{ and } F_2 = \begin{pmatrix} a_{11} & 0 & a_{13} & 0 & 0 \\ 0 & a_{22} & a_{23} & 0 & 0 \\ 0 & 0 & a_{33} & 0 & 0 \\ 0 & 0 & a_{43} & a_{44} & 0 \\ 0 & 0 & a_{53} & 0 & a_{55} \end{pmatrix},$$

for which the index of nilpotency of the triple commutator is also 2.

The proof is straightforward as in $[F_{11}, F_{12}, F_{13}] = (a_{ij})$ nonzero are only the elements $a_{31}, a_{32}, a_{34}, a_{35}$ while in $[F_{21}, F_{22}, F_{23}] = (b_{ij})$ such are the elements $b_{13}, b_{23}, b_{43}, b_{53}$.

Corrolary 3.2 *The two classes of algebras presented by the matrices of type F_1 and F_2 satisfy the identities defining the varieties $\Re_3$, $\Re_4$ and $\Re_5$ and consequences of the identities defining $\Re_1$ and $\Re_2$, but they do not belong to neither of these five varieties.*

Proof: We'll consider only matrices of type F_1 as for those of type F_2 everything is done in an analogous way.

a/ The matrices of type F_1 satisfy the identity $[F_{11}, F_{12}, F_{13}]^2 = 0$, which is a consequence of the identities defining $\Re_1$ and $\Re_2$.

b/ In $[F_{11}, F_{12}, F_{13}]F_1 = (a_{ij})$ the elements $a_{31}, a_{32}, a_{34}, a_{35}$ are nonzero and in $F_1[F_{11}, F_{12}, F_{13}] = (b_{ij})$ the elements $b_{31}, b_{32}, b_{34}, b_{35}$ are nonzero as well. Thus the identities defining either $\Re_1$ or $\Re_2$ are not satisfied.

The considered matrices obviously satisfy the identities defining $\Re_4$ and $\Re_5$ but these matrices satisfy identities of lower degree as well.

The same is valid for the variety $\Re_3$ as in both the matrices $[F_{11}, F_{12}, F_{13}] = (c_{ij})$ and $F_1[F_{11}, F_{12}, F_{13}] = (b_{ij})$ the only nonzero elements are $c_{31}, c_{32}, c_{34}, c_{35}$ and $b_{31}, b_{32}, b_{34}, b_{35}$ and this gives that the identity defining $\Re_3$ is satisfied.

This ends the proof.

For completeness we give the graphical representation of these two classes, namely

$$\begin{array}{ccccc} \times & & & & \\ & \circ & & & \\ \times & \circ & \bullet & * & \div \\ & & & * & \\ & & & & \div \end{array} \text{ and } \begin{array}{ccccc} \times & & \times & & \\ & \circ & \circ & & \\ & & \bullet & & \\ & & * & * & \\ & & \div & & \div \end{array}$$

, respectively, but it is not more informative than the standard one.

It is tempting to consider two more types of matrices, namely

$$\begin{pmatrix} 0 & 0 & 0 & 0 & a_{15} \\ 0 & 0 & 0 & a_{24} & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ 0 & a_{42} & 0 & 0 & 0 \\ a_{51} & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & a_{13} & 0 & a_{15} \\ 0 & 0 & a_{23} & a_{24} & 0 \\ 0 & 0 & a_{33} & 0 & 0 \\ 0 & a_{42} & a_{43} & 0 & 0 \\ a_{51} & 0 & a_{53} & 0 & 0 \end{pmatrix}$$

, but they are not closed due to the multiplication rule. The problem comes from the fact that here we work with the second diagonal.

This shows the importance of paying attention the types of matrices under consideration to form matrix subalgebras.

As a conclusion further work on the topic could be proposed:

1.To find matrix algebras over the Grassmann algebra with symmetric identities of minimal degree like

$$U_1[X_1,Y_1,Z_1][X_2,Y_2,Z_2]U_2=0$$

or $U_1[X_1,Y_1,Z_1]U_2[X_2,Y_2,Z_2]U_3=0$;

2.To find matrix algebras over the Grassmann algebra with symmetric identities of minimal degree like

$U_1[X_1,Y_1,Z_1]^k U_2=0$ for small $k\geq 1$

or $U_1[X_1,Y_1,Z_1]U_2[X_2,Y_2,Z_2]U_3\cdots[X_s,Y_s,Z_s]U_{s+1}=0$ for $s>2$;

3.To classify PI-algebras with minimal identities of type

$$U_1^{k_1}[X_1,Y_1,Z_1]U_2^{k_2}[X_2,Y_2,Z_2]...U_{s-1}^{k_{s-1}}[X_l,Y_l,Z_l]U_s^{k_s}=0.$$

4. THE PURPOSE OF DRAWING PI-ALGEBRAS

In the previous section we mentioned possibilities for drawing PI-matrix algebras. It seems a bit unusual to share proving theorems in PI-theory with drawing special types of matrices and discussing different ways of interpreting the way of drawing. But the reason to show the applicability of mathematical ideas in everyday life leads to interest, usefulness and importance combined together.

Now we speak about innovations, new technologies, digital abilities in order to face the new chalenges of the digital world all over around us. This is really important and at the same time any goal can't be the centre of the universe. Parallely with reaching it we have to encourage thinking, understanding and using the results of these human qualities to find new knowledge, not only competence, to develop our brains and equip them with features leading to actions towards human progress.

How drawing PI-theory could help such a process?

a/ The proof of the different identities discussed could be used in a proper way to give technics to students to deal with matrix algebra for example. If we replace a problem of type:

Find the expression $A^2-2AB+3E$ **for given matrices** A,B with

Prove that $[A,B]^2=0$ **for any two matrices of a given type** (not specified here) or

Find all couples of matrices A,B **satisfying the identity** $[A,B]^2=0$,

we define a class of algebras with a given property and could go back to it in our work with the students later to discuss it.Thus we generalize the rules for the different operations with matrices on a higher, qualitative, level.

It is only one example that not the fact is the most important but what we could reach using it. Example 2.1 could give more ideas in this direction.

b/ Introducing matrices at school and drawing them with GeoGebra for example could be simultaneously a challenge, education and fun.

c/ The forms of the different matrices could be considered as constructing blocks to show symmetry or asymmetry in pictures, fabrics, mosaics, etc. We could propose its application when doing fitness as well as different positions of our body.

d/ Checking closeness or not of multiplication for different types of matrices we could classify them on this base and show the advantage of ones over others and thus showing an easier way to introduce the notion of subalgebras.

e/ An identity for a class of matrix algebras could be considered as a feature of a group of people thus giving a new way of doing classification needed by more important reasons.Thus a consequence of an identity will be thought as an additional feature defined by a more basic one. Just two examples: what could we expect from children being without control after school or from people with the ability to write comfortably with both their hands? The theory of finding identities and describing all its consequences could be the framework of the approach for many specific cases no matter of their nature. It could give a logical and investigational view on the world around us.

f/ Let consider a situation in PI-theory that a class of algebras does not satisfy a special identity. How could we interprete it? A group of people is not authorized to act in a special way. Something is allowed, something is forbidden. Something is possible to be done, something else not as no information is available for the latter.

g/ We could go to manifacture as well. The forms of different kids' toys could ressemble the forms of the different types of matrices investigated in PI-theory. A correspondence between the different factors of an identity and different types of lamps in the toys could be considered as possible. The Grassmann identity $[x_1, x_2, x_3] = 0$ will mean a quadratic lamp for example sparkling once, k sparkings will correspond to $[x_1, x_2, x_3]^k = 0$, while $x^s = 0$ will indicate s sparkings of a round lamp.

Our fantasy could go on. But it is not really a fantasy. It is a way to show how people could be encouraged to think, to come to conclusions and to organize their life in harmony with their human nature.

We end our survey with the conviction that a progress in human understanding via education could be reached if we follow a path built by the following elements – exposing facts, explanation of facts, giving an application of facts, forming an ideology of facts in order to come naturally to the conclusion of each individual that such paths are the fastest highways in our life.

Achieving this conviction is the everlasting goal of science and the science community has to have the ambition to give his contribution to it.

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РАЗНООБРАЗНИ АСПЕКТИ НА ТЕОРИЯТА НА ПОЛИНОМНИТЕ ТЪЖДЕСТВА

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Резюме: *Ако разглеждаме Алгебрата като карта и историческото й развитие като вертикални пътища по нея с имена (например **A**(алгебрични структури), **F**(теория на полето), **C**(комутативна алгебра), **AS**(асоциативни пръстени и алгебри), **G**(теория на групите) и т.н., то тези пътища биха имали най-вече вертикални разклонения при задълбочаване изследванията по дадена тема. Това изследване дава пример както за такива вертикални разклонения, така и за възможен хоризонтален маршрут започващ от инволюции в матрични алгебри и водещ към друг подход към комплексните числа, правейки в същото време малки крачки и във вертикална посока.*

Разглеждат се и някои подалгебри на матричната $n \times n$ алгебра над Грасмановата алгебра и се изучават тъждествата от ниска степен за тях. За конкретни n тези подалгебри са илюстрирани графично и е отбелязана възможността за използване и на системата Geogebra за тази цел. Обърнато е внимание и на някои приложения при разглеждания подход, стигайки до нов аспект на класическата теория на PI-алгебрите.

Ключови думи: *Инволюция, симетрични и антисиметрични елементи, φ-полиномни тъждества, Грасманова алгебра, матрични алгебри с Грасманови елементи, многообразия от алгебри, стандартен полином, GeoGebra.*

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