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"MATHEMATICS, INFORMATICS AND PHYSICS"

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"MATHEMATICS, INFORMATICS AND PHYSICS"

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Physics

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THE T - IDEAL OF THE X - FIGURAL MATRIX ALGEBRA

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Abstract: *In the paper we consider the X - figural matrix algebra and describe the T-ideal of its identities over the infinite dimensional Grassmann algebra E.*

Keywords: *Grassmann algebra, matrix algebras with Grassmann entries, identities, finitely generated T-ideals*

PRELIMINARIES

We recall the definition of the infinite dimensional Grassmann algebra E as

$$E = E(V) = K\langle e_1, e_2, \ldots \mid e_i e_j + e_j e_i = 0 i, j = 1,2,\ldots \rangle,$$

where the field K has characteristic zero.

The algebra E is in focus of recent research in PI-theory. Its importance is connected with the structure theory for the T-ideals of identities of associative algebras developed by Kemer [4]. Many other applications of E are investigated as well, see for example [7].

The significance of considering the matrix algebra $M_n(E)$ is confirmed by the following statement as the trivial isomorphism $E \otimes M_n(K) \cong M_n(E)$ holds:

Proposition 1 [3, Corollary 8.2.4] *For every PI-algebra R there exists a positive n such that $T(R) \supseteq T(M_n(E))$, i.e. R satisfies all polynomial identities of the $n \times n$ matrix algebra $M_n(E)$ with entries from the Grassmann algebra.*

For the PI-properties of E and $M_n(E)$ one could see [2,5]. Here we formulate:

Proposition 2 [5, Corollary, p. 437] *The T-ideal $Id(E)$ is generated by the identity* $[x_1, x_2, x_3] = 0$.

Proposition 3 [2, Lemma 6.1] *The algebra E satisfies $S_n(x_1, \ldots, x_n)^k = 0$ for all $n, k \geq 2$ and*

$$S_n(x_1, \ldots, x_n) = \sum_{\sigma \in Sym(n)} (-1)^{\sigma} x_{\sigma(1)} \ldots x_{\sigma(n)}$$

being the standard identity.

Proposition 4 [2, Corollary 6.6] *The algebra $M_n(E)$ does not satisfy the identity*

$$S_m(x_1, \ldots, x_m)^n = 0$$

for any m.

It is an open question [1, p.356] to describe the identities of minimal degree of $M_n(E)$. Even for $n = 2$ we know very little. There is a result of U. Vishne [8] that the minimal degree of an identity for $M_2(E)$ is 8 and he gives the explicit form of two concrete multilinear polynomials $T_1(x_1, \ldots, x_6; y_1, y_2)$ and $T_2(x_1, \ldots, x_5; y_1, y_2, y_3)$ being identities for $M_2(E)$. We rely on [8] for their definition.

In the paper we find some identities of minimal degree for the so called X -figural matrix algebra and describe completely its T-ideal in both the cases (odd and even order of the considered matrices).

THE X - FIGURAL MATRIX ALGEBRA

We use the term " X -figural matrix algebra" for algebras of the type described below. We consider two cases if the order of the matrices is odd or even.

Let $R1(E)$ be the X -figural $(2n+1)\times(2n+1)$ matrix algebra of the matrices of type

$$\begin{pmatrix} a_1 & 0 & ... & ... & ... & 0 & a_1 \\ 0 & a_2 & 0 & ... & 0 & a_2 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & ... & 0 & a_{n+1} & 0 & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & a_{2n} & 0 & ... & 0 & a_{2n} & 0 \\ a_{2n+1} & 0 & ... & ... & ... & 0 & a_{2n+1} \end{pmatrix} : a_j \in E, j = 1,...,2n+1.$$

The algebra $R1(E)$ is with basis

$$\begin{array}{lcl} f_1 & = & e_{11} + e_{1,2n+1} \\ f_2 & = & e_{22} + e_{2,2n} \\ ... & ... & ... \\ f_n & = & e_{nn} + e_{n,n+2} \\ f_{n+1} & = & e_{n+1,n+1} \\ f_{n+2} & = & e_{n+2,n} + e_{n+2,n+2} \\ ... & ... & ... \\ f_{2n+1} & = & e_{2n+1,1} + e_{2n+1,2n+1} \end{array} .$$

This basis is a multiplicative one with properties $f_i f_j = f_i$ for $i = j$ or $i + j = 2n + 2$ and $f_i f_j = 0$ for $i \neq j$ or $i + j \neq 2n + 2$.

We consider the isomorphism:

$$f_1 \approx e_{2n+1,2n+1}, f_2 \approx e_{2n,2n},..., f_n \approx e_{n+2,n+2}, f_{n+1} \approx e_{n+1,n+1},$$
$$f_1 - f_{2n+1} \approx e_{1,2n+1}, f_2 - f_{2n} \approx e_{1,2n},...,$$
$$f_{n-1} - f_{n+3} \approx e_{1,n+3}, f_n - f_{n+2} \approx e_{1,n+2}.$$

Thus $R1(E)$ is PI-equivalent to the algebra of the matrices of type

$$\begin{pmatrix} 0 & \dots & \dots & 0 & E & \dots & E \\ & \underbrace{\ddots}_{n-2} & \vdots & \vdots & 0 & \dots & 0 \\ & & 0 & & & & \\ \vdots & & & E & & & \vdots \\ & & & & \ddots & & \\ & & & & & \underbrace{\ddots}_{n-1} & \\ 0 & & & \cdots & & 0 & E \end{pmatrix}.$$

The even case gives the algebra $R2(E)$ of the matrices of type

$$\begin{pmatrix} a_1 & 0 & \dots & \dots & \dots & \dots & 0 & a_1 \\ 0 & a_2 & 0 & \dots & \dots & 0 & a_2 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & a_n & a_n & 0 & \dots & 0 \\ 0 & \dots & 0 & a_{n+1} & a_{n+1} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & a_{2n-1} & 0 & \dots & \dots & 0 & a_{2n-1} & 0 \\ a_{2n} & 0 & \dots & \dots & \dots & \dots & 0 & a_{2n} \end{pmatrix}; a_j \in E, j = 1,\dots,2n.$$

It is with basis

$$\begin{array}{lll} f_1 = e_{11} + e_{1,2n}, & f_2 = e_{22} + e_{2,2n-1}, & \dots, \\ f_n = e_{nn} + e_{n,n+1}, & f_{n+1} = e_{n+1,n} + e_{n+1,n+1}, & \\ f_{n+2} = e_{n+2,n-1} + e_{n+2,n+2}, & f_{n+3} = e_{n+3,n-2} + e_{n+3,n+3}, & \dots, \\ f_{2n-1} = e_{2n-1,2} + e_{2n-1,2n-1}, & f_{2n} = e_{2n,1} + e_{2n,2n}, & \end{array}$$

where $f_i f_j = f_i$ for $i + j = 2n+1$ and $f_i f_j = 0$ for $i + j \neq 2n+1$.

Analogously

$$f_1 \approx e_{2n,2n}, f_2 \approx e_{2n-1,2n-1},\dots,$$
$$f_{n-1} \approx e_{n+2,n+2}, f_n \approx e_{n+1,n+1}, f_{n+1} \approx e_{nn},$$
$$f_1 - f_{2n} \approx e_{1,2n}, f_2 - f_{2n-1} \approx e_{1,2n-1},\dots,$$
$$f_{n-1} - f_{n+2} \approx e_{1,n+2}, f_n - f_{n+1} \approx e_{1,n+1}.$$

We get that the algebra $R2(E)$ is PI-equivalent to the algebra of the matrices

$$\begin{pmatrix} 0 & \ldots & 0 & 0 & * & * \\ & \underbrace{\ddots}_{n-3} & & & & 0 \\ \vdots & & 0 & & & \vdots \\ \vdots & & & E & & \vdots \\ & & & & \underbrace{\ddots}_{n-1} & 0 \\ 0 & & \cdots & & 0 & E \end{pmatrix}.$$

PI-PROPERTIES

We start considering the two X-figural algebras with entries from a field K with characteristic zero.

Theorem 1 *The algebras* $R1(K)$ *and* $R2(K)$ *satisfy the identity*

$$X[X_1, X_2] = 0.$$

Proof: It is easy to be seen that the sum of the entries in each column of the commutator of two matrices for any of the two algebras is zero. Thus the multiplication rule for $X[X_1, X_2]$ gives the desired identity.

Remark 1 It could be proved that the T-ideal of any of the algebras $R1(K)$ and $R1(K)$ is generated by the identity $X[X_1, X_2] = 0.$ The needed considerations are generalized in the proof of Theorem 4.

Let now the entries of the two matrices be Grassmann entries. In this case the sum of the entries in each column of the matrix $[X_1, X_2, X_3]$ is zero and we get

Theorem 2 *The algebras* $R1(E)$ *and* $R2(E)$ *satisfy the identity*

$$X[X_1, X_2, X_3] = 0.$$

In [6] the following theorem was proved:

Theorem 3 *Any algebra, satisfying the identity* $X[X_1, X_2, X_3] = 0$*, satisfies the identities* $T_1(x_1,...,x_6; y_1, y_2) = 0$ *and* $T_2(x_1,...,x_5; y_1, y_2, y_3) = 0$*, defined in* [8].

Thus we get

Corollary 1 *In the algebras* $R1(E)$ *and* $R2(E)$ *the identities*

$T_1(x_1,...,x_6; y_1, y_2) = 0$ and $T_2(x_1,...,x_5; y_1, y_2, y_3) = 0$ hold.

For any polynomial $f \in K\langle X \rangle$ the values of f in $R1(E)$ and in E are related as follows: if $f|_{R1(E)} = (a_{ij})$, then the entry $a_{n+1,n+1} = f|_E$.

Thus we come to

Corollary 2 $Id(R1(E)) \subset Id(E).$

An analogous inclusion is valid for the algebra $R2(E)$ as well:

Corollary 3 $Id(R2(E)) \subset Id(E).$

Proof: We consider the algebra $R_0(E) = \{R2(E) : a_i = 0, i = n+1,...,2n\}$ and obviously $R_0(E) \approx E$. As $R_0(E) \subset R2(E)$ we get that

$$Id(R2(E)) \subset Id(R_0(E)) = Id(E).$$

We could describe the T-ideals of the algebras $R1(E)$ and $R2(E)$ completely. They turned to be finitely generated.

Theorem 4 *The T-ideals $Id(R1(E))$ and $Id(R2(E))$ are generated by the identity*

$$X[X_1, X_2, X_3] = 0.$$

Proof: We give the proof in the odd case only. For the even case only slightly changes of the indices are needed ($\ldots_{2n+1} = \ldots_{2n}$)

As all identities in E follow from $[x_1, x_2, x_3] = 0$, we have to show that there are no consequences of this identity which are identities in $R1(E)$ but are not consequences of $x[x_1, x_2, x_3] = 0$.

I. We start with the consequences of degree 4. We get them in two ways - multiplying $[x_1, x_2, x_3] = 0$ on either side by x or by substituting in the identity any of the variables i.e. $x_i = xx_i$ (the substitution $x_i = x_i x$ leads to changing only the indices of the variables).

In the transformations below we repeatedly use the trivial identity

$$[a, bv] = [a, b]v + b[a, v].$$

We get

$$\begin{aligned}
[x_1, x_2, x_3 x] &= [x_1, x_2, x_3]x + x_3[x_1, x_2, x] \\
[x_1, x_2 x, x_3] &= [[x_1, x_2]x, x_3] + [x_2[x_1, x], x_3] \\
&= -[x_3, [x_1, x_2]x] - [x_3, x_2[x_1, x]] \\
&= -[x_1, x_2][x_3, x] - [x_3, [x_1, x_2]]x \\
&\quad - x_2[x_3, [x_1, x]] - [x_3, x_2][x_1, x].
\end{aligned}$$

As in $R1(E)$ the identity $X[X_1, X_2, X_3] = 0$ holds, the possible identity of degree 4 will be of the form

$$T = \alpha[X_1, X_2, X_3]X + \beta([X_1, X_2][X_3, X] + [X_3, X_2][X_1, X])$$

for some values of $\alpha, \beta \in K$.

We follow the $(1,1)$ entry of the matrix T in $R1(E)$.

We use the notations $X_1 = (a_i)$, $X_2 = (b_i)$, $X_3 = (c_i)$ and $X = (d_i)$.

The $(1,1)$-entry of $[X_1, X_2]$ is $[a_1, b_1] + a_1 b_{2n+1} - b_1 a_{2n+1}$. Modulo the Grassmann identity the $(1,1)$-entry of $[X_1, X_2, X_3]X$ is

$$\begin{aligned}
&([a_1 b_{2n+1} - b_1 a_{2n+1}, c_1] + ([a_1, b_1] + a_1 b_{2n+1} - b_1 a_{2n+1})c_{2n+1} \\
&- c_1([a_{2n+1}, b_{2n+1}] + a_{2n+1} b_1 - b_{2n+1} a_1))(d_1 + d_{2n+1}).
\end{aligned}$$

For the $(1,1)$ entry of $[X_1, X_2][X_3, X] + [X_3, X_2][X_1, X]$ we get

$$\begin{aligned}
&-([a_1, b_1] + a_1 b_{2n+1} - b_1 a_{2n+1})[d_1 + d_{2n+1}, c_1 + c_{2n+1}] \\
&-([c_1, b_1] + c_1 b_{2n+1} - b_1 c_{2n+1})[d_1 + d_{2n+1}, a_1 + a_{2n+1}].
\end{aligned}$$

We calculate the polynomial T for matrices X_1, X_2, X_3, X in which the values of the corresponding entries are the following elements of E, namely

$$\begin{array}{llll}
a_1 = e_1 & b_1 = e_2 & c_1 = e_3 & d_1 = e_2 e_3 \\
a_{2n+1} = e_1 & b_{2n+1} = e_4 & c_{2n+1} = e_5 & d_{2n+1} = e_1 e_6.
\end{array}$$

Thus we come to

$$\alpha e_1 e_2 e_3 e_4 e_5 + \beta.0 = 0$$

meaning that $\alpha = 0$.

Really the form of the $(1,1)$ entry of $[X_1,X_2][X_3,X]+[X_3,X_2][X_1,X]$ shows that if for every i either d_i or any of c_i and a_i are of even degree, this entry is zero. However we have to see the value of the $(1,1)$-entry of $[X_1,X_2,X_3]X$ as well.

Next we make the substitution

$$\begin{array}{llll} a_1=e_1 & b_1=e_3 & c_1=e_5 & d_1=e_7 \\ a_{2n+1}=e_2 & b_{2n+1}=e_4 & c_{2n+1}=e_6 & d_{2n+1}=e_8. \end{array}$$

The result is

$$\begin{aligned} &2\beta(-e_1e_3e_6e_7-e_1e_3e_6e_8-e_1e_4e_6e_7-e_1e_4e_6e_8 \\ &+e_2e_3e_5e_7+e_2e_3e_5e_8+e_2e_4e_5e_7+e_2e_4e_5e_8)=0. \end{aligned}$$

This means that $\beta=0$ and in $R1(E)$ there are no identities of degree 4 which are consequences of $[X_1,X_2,X_3]=0$.

II. It is enough to consider only the identities of degree 4 as for the consequences of higher degree we have the following recurrent relations in $R1(E)$:

$$\begin{aligned} [x_1,x_2,x_3u_1u_2...u_{n-1}u_n] &= [x_1,x_2,x_3u_1u_2...u_{n-1}]u_n+x_3(u_1u_2...u_{n-1})[x_1,x_2,u_n] \\ &= [x_1,x_2,x_3u_1u_2...u_{n-1}]u_n \\ [x_1,x_2v_1v_2...v_{k-1}v_k,x_3] &= [x_1,x_2v_1v_2...v_{k-1},x_3]v_k \\ &\quad -[x_1,x_2v_1v_2...v_{k-1}][x_3,v_k]-[x_3,x_2v_1v_2...v_{k-1}][x_1,v_k]. \end{aligned}$$

Thus we get that $Id(R1(E))=\langle X[X_1,X_2,X_3]=0\rangle$.

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T-ИДЕАЛЪТ НА X - ФИГУРАЛНАТА МАТРИЧНА АЛГЕБРА

Цецка Рашкова

Русенски университет "Ангел Кънчев"

***Резюме**: В статията се разглежда X -фигуралната матрична алгебра и се описва T -идеалът на тъждествата в нея в случая когато тя се разглежда над безкрайномерната Грасманова алгебра E.*

***Ключови думи:** Грасманова алгебра, матрични алгебри с Грасманови елементи, тъждества, крайнопородени T -идеали.*

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