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TEACHING GROUP THEORY VIA TRANSFORMATIONS[2]

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Abstract: *The paper gives an approach for achieving better understanding the basic ideas of Group Theory, a part of an university course in Algebra at the University of Ruse, giving more light on transformations and using software for visualization of group properties.*
Keywords: *general and special linear groups, elements of finite order, transformations, visualization*

INTRODUCTION

The nowadays situation with the background of the first year students forces the teaching of the university courses to be in close connection with this backbround. We think that better understanding of the material could be facilitated if more different kinds of examples prevail in the exposition and the knowledge from earlier taught courses is widely used.

The paper gives an approach for achieving this goal concerning the basic ideas of Group Theory, a part of an university course in Algebra at the University of Ruse.

The review is based partially on [1,2,3,4,5].

Usually we illustrate the taught material with examples of groups whose elements are different kinds of numbers. The symmetric group S_n is illustrated usually working mostly in S_3. But rarely we connect it with the symmetries of a triangle. As a whole the transformations as group elements are not included in the exposition especially in short courses on the topic. Sometimes we mention the general linear group GL_n but usually we do not work with its elements illustrating group properties. As courses in Geometry and Linear Algebra usually precede the course in Algebra it is successful to use transformations of lines in the plane and of surfaces in the space to make clearer what the elements of the general linear group are really like.

We recall that the general linear group $GL_n(K)$ consists of the nonsingular $n\times n$-matrices with entries from a field K, the special linear group $SL_n(K)$ consists of the $n\times n$-matrices with determinant equal to 1, the general orthogonal group $GO_n(K)$ consists of the $n\times n$-matrices A with the property $A^t = A^{-1}$ and $SO_n(K)$ is its subgroup of matrices with determinant equal to 1.

THE GROUP $GL_2(R)$ AND ITS SUBGROUPS

Any matrix $A_{2,2} = (a_{ij})$ defines a transformation $g_A \to \left| \begin{array}{l} x^* = a_{11}x + a_{12}y \\ y^* = a_{21}x + a_{22}y \end{array} \right.$ of the plane.

Let consider for example the matrix $A = \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix}$ with the corresponding

[2]Partially supported by Grant 2015-FPNO-02 at the Angel Kanchev University of Ruse.

transformation g_A. It is easily seen that A is of infinite order. However we could illustrate $g_A^i(l_j)$ on some lines l_j for small i. Starting with $l_1 \to y = x$ as $x = \dfrac{3x^* + 2y^*}{3}, y = \dfrac{y^*}{3}$ we get consequently $g_A(l_1) \to y^* = -3x^*$, $g_A^2(l_1) \to y^{**} = -\dfrac{9}{7}x^{**}$, $g_A^3(l_1) \to y^{***} = -\dfrac{27}{25}x^{***}$,

By the system for computer algebra *Mathematica* we visualize these transfomations on the following Figure1:

Plot[{x,-9x/7,-27x/25},{x,-3,3}]

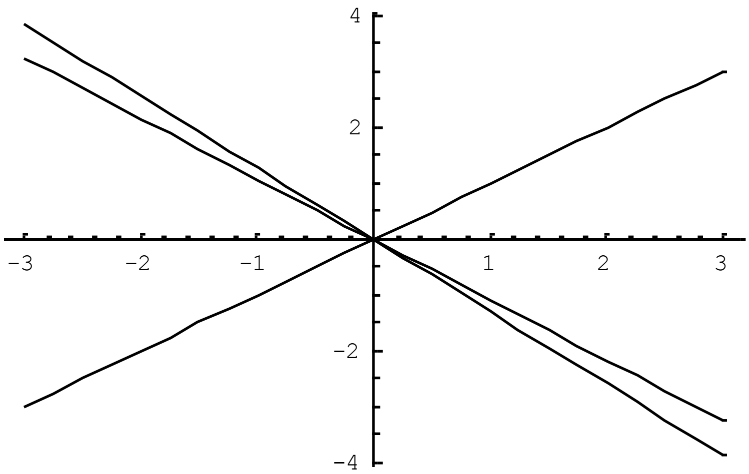

Fig.1

It is interesting to note that there are lines in the plane which are not changed via the transformation g_A. For the line $l_2 \to y = -x$ we get that

$$g_A(l_2) = g_A^2(l_2) = ... = g_A^k(l_2) = l_2 \to y = -x, k \in N.$$

We could mach special elements of the group $GL_2(R)$ defining transformations of the plane such as expansions and compressions:

a/ The matrix $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$ is an element of infinite order and it “stretches” the vector $\begin{pmatrix} x \\ y \end{pmatrix}$ along the x-axis to $\begin{pmatrix} kx \\ y \end{pmatrix}$ for $k > 1$ and “compresses” it along the x-axis for $0 < k < 1$;

b/ Similarly the matrix $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$ stretches or compresses the vector $\begin{pmatrix} x \\ y \end{pmatrix}$ to $\begin{pmatrix} x \\ ky \end{pmatrix}$ along the y-axis.

Now we consider elements of the subgroup $SL_2(R)$.

Let g_B correspond to the matrix $B = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$., an element of order 3. It defines the following transformation: $x^{*}=-y$, $y^{*}=x-y$. Thus $x=-x^{*}+y^{*}$ and $y=-x^{*}$.

Let consider again the line $l_1 \to y=x$. It is easy to see that

$g_B(l_1) \to y^{*}=0,\ g_B^2(l_1) \to x^{**}=0,\ g_B^3(l_1) \to y^{***}=x^{***}$.

Using the system *Mathematica* we visualize the situation in Figure 2.

Plot[{x,0,-x},{x,-3,3}]

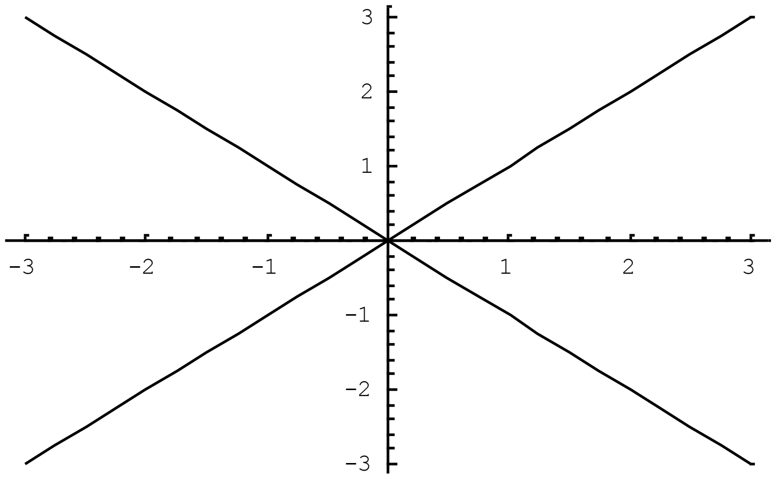

Fig.2

We visualize the transformation of a part of the line $x^2+y^2=1$ via the same matrix B as well. Applying g_B to the line $l_3 \to y=\sqrt{1-x^2}$, we get consequently

$$y=\sqrt{1-x^2} \to y^{*}=x^{*}\pm\sqrt{1-(x^{*})^2} \to y^{**}=\frac{x^{**}\pm\sqrt{2-(x^{**})^2}}{2}.$$

The corresponding visualization is on Figure 3:

Plot[{Sqrt[1-x^2],x+Sqrt[1-x^2],(x+Sqrt[2-x^2])/2},{x,-0.9,0.9}]

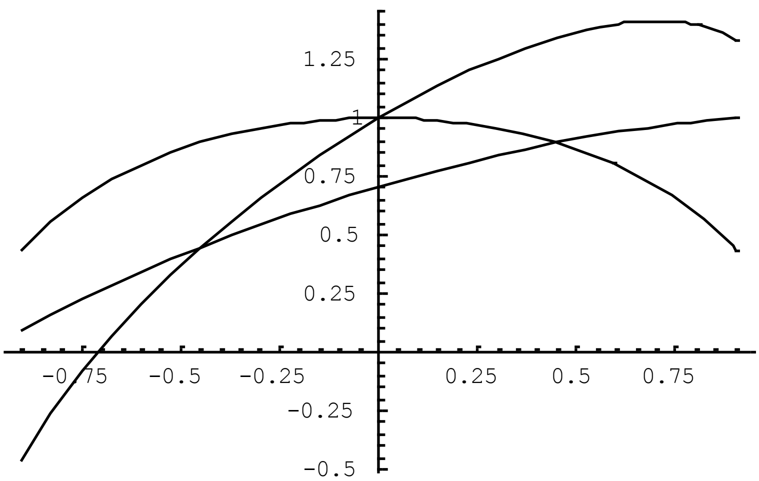

Fig.3

For $l_4 \to y = x^2$ we get $g_B(l_4) \to y^* = x^* \pm \sqrt{-x^*}$,

$g_B^2(l_4) \to y^{**} = \dfrac{-1 \pm \sqrt{1+4x^{**}}}{2}$, $g_B^3(l_4) \to y^{***} = (x^{***})^2$.

Mathematica gives the visualization on Figure 4:

Plot[{x^2,x+Sqrt[-x],(-1+Sqrt[1+4x])/2},{x,-0.25,0}]

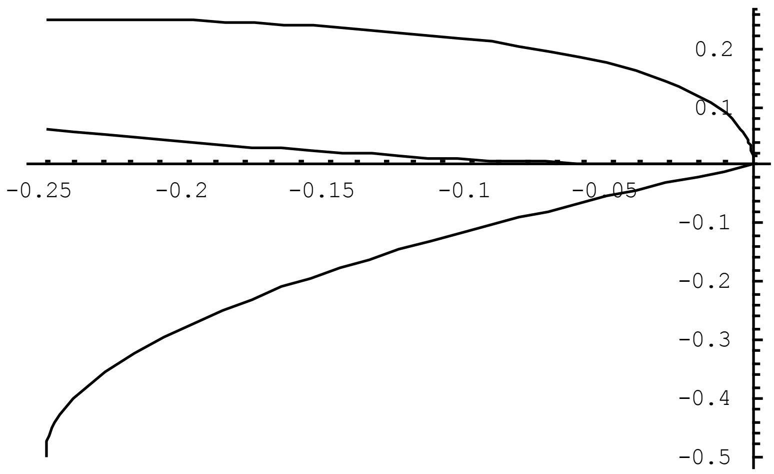

Fig.4

For the forth order element g_C, where $C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $l_1 \to y = x$ we have

$g_C(l_1) \to y^* = -x^*,\ g_C^2(l_1) \to y^{**} = x^{**},$
$g_C^3(l_1) \to y^{***} = -x^{***},\ g_C^4(l_1) \to y^{****} = x^{****}$.

Considering $l_4 \to y = x^2$ we get

$g_C(l_4) \to (y^*)^2 = -x^*,\ g_C^2(l_4) \to y^{**} = -(x^{**})^2,$
$g_C^3(l_4) \to (y^{***})^2 = x^{***},\ g_C^4(l_4) \to y^{****} = (x^{****})^2$.

Dealing with the subgroup $SL_2(R)$ the infinite order element $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, taking $\begin{pmatrix} x \\ y \end{pmatrix}$ to $\begin{pmatrix} x+ky \\ y \end{pmatrix}$, defines a shear in the x-direction. Respectively the matrix $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$, taking $\begin{pmatrix} x \\ y \end{pmatrix}$ to $\begin{pmatrix} x \\ y+kx \end{pmatrix}$, defines a shear in the y-direction.

We could mach special elements of the orthogonal group $GO_2(R)$ as well corresponding to basic transformations of the plane:

a/ The matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is an element of second order and it defines reflection towards the x-axis transforming a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ into a vector $\begin{pmatrix} x \\ -y \end{pmatrix}$;

b/ The matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is an element of second order and it defines reflection towards the y-axis transforming a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ into a vector $\begin{pmatrix} -x \\ y \end{pmatrix}$;

c/ The matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is an element of second order and it defines reflection towards the line $y = x$ transforming a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ into a vector $\begin{pmatrix} y \\ x \end{pmatrix}$.

A more detailed study could be made on the special orthogonal group $SO_2(R)$ as well**.** Finding its elements could be an individual students' work. As a result we come to the matrix $A_n = \begin{pmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{pmatrix}$ of the rotation g_{A_n} on an angle $2\pi/n$ and the explanation of the other name of $SO_2(R)$ **-** the group of rotations.

The action of $SO_2(R)$ on different lines in the plane could be investigated by the students. An individual students' work could include for example the proof of the following basic group properties:

1/ The group $SO_2(R)$ is a normal subgroup of $GO_2(R)$;

2/ The group $SL_2(R)$ is a normal subgroup of $Gl_2(R)$.

THE GROUP $GL_3(R)$ AND ITS SUBGROUPS

As an example we consider the transformation g_P, where $P = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. As $P^2 = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$, obviously $g_P \in SO_3(R)$ is of infinite order.

Let consider the surface $\alpha_1 \to z = x^3 + y$. We give in Figure 5 Its visualization:

Plot3D[x^3+y,{x,-3,3},{y,-3,3}]

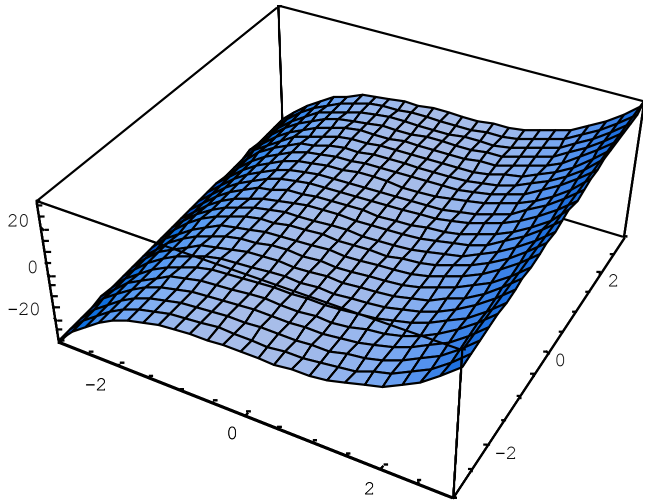

Fig.5

As $x^* = x + y, y^* = y, z^* = y + z$ we get $x = x^* - y^*, y = y^*, z = -y^* + z^*$. Thus the surface $z = x^3 + y$ is transformed into the surface $z^* = (x^* - y^*)^3 + 2y^*$. *Mathematica* visualizes it in Figure 6:

Plot3D[(x-y)^3+2y,{x,-3,3},{y,-3,3}]

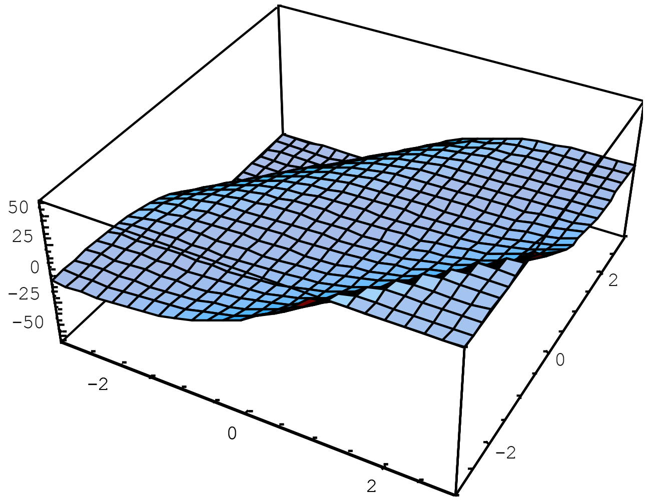

Fig.6

Now Figures 7 and 8 visualize consequently $\alpha_2 \to z = xy$ and $g_P(\alpha_2) \to z^* = y^*(x^* - y^* + 1)$:

Plot3D[(xy,{x,-3,3},{y,-3,3}]

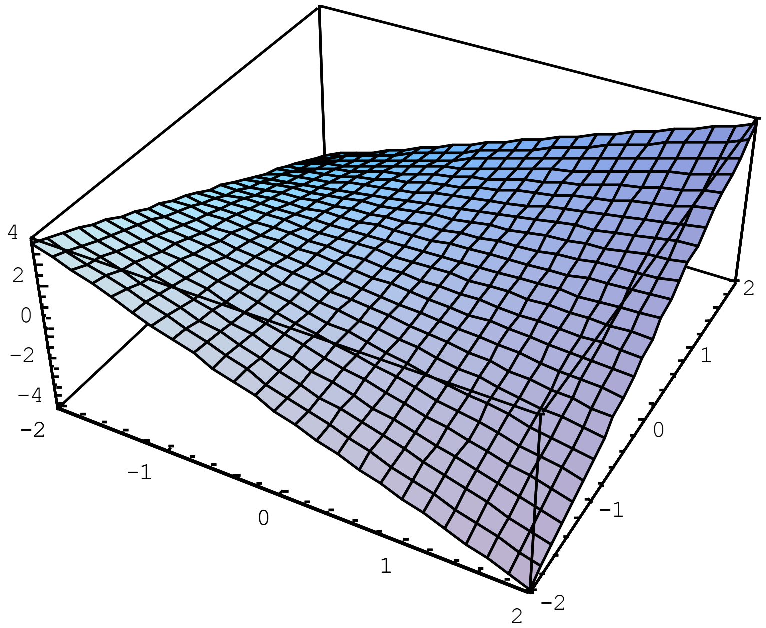

Fig.7

Plot3D[y(x-y+1),{x,-3,3},{y,-3,3}]

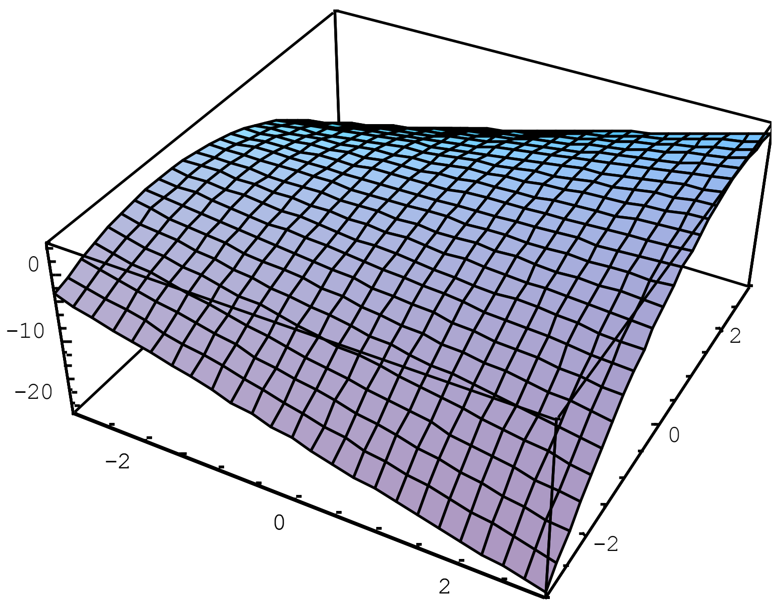

Fig.8

It is interesting to note that there are surfaces β_i such that $g_P(\beta_i)=\beta_i$. One example is $\beta_1 \to z = x + y^2$.

At the end we consider the element $g_Q \in SL_3(R)$ of order 3 for $Q=\begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Applying it to the surface $\alpha_1 \to z = x^3 + y$ we get

$$g_Q(\alpha_1) \to z^* = (y^* - x^*)^3 - x^*,\ \ g_Q^2(\alpha_1) \to z^{**} = -(y^{**})^3 + x^{**} - y^{**}.$$

We illustrate $g_Q(\alpha_1)$ and $g_Q^2(\alpha_1)$ by *Mathematica* on Figures 9 and 10:

Plot3D[(y-x)^3-x, {x,-3,3}, {y,-3,3}]

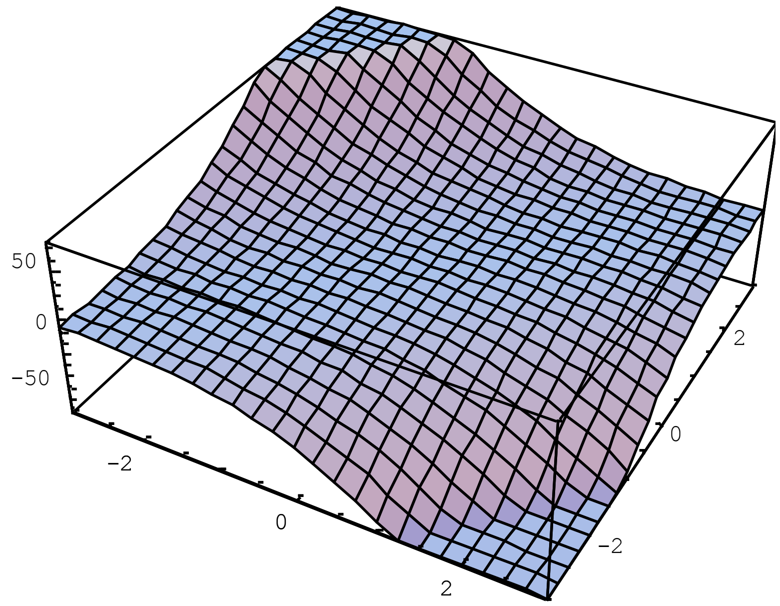

Fig.9

Plot3D[-y^3+x-y, {x,-3,3}, {y,-3,3}]

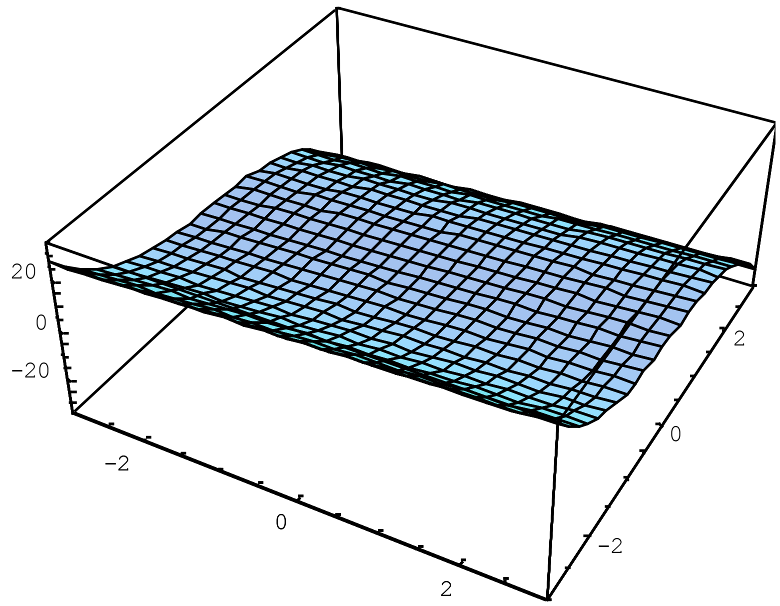

Fig.10

More light could be given to the inverse of the finite order elements as well. In the last case we could describe $g_Q^{-1} = g_Q^2$ with its action on the surface $\alpha_1 \to z = x^3 + y$.

THE SYMMETRY GROUP OF A SQUARE

Consider the square with vertices a,b,c,d:

The notation (a,b) means that the vertex a is transformed to the vertex b. The two mappings $A=\{(a,b),(b,c),(c,d),(d,a)\}$ and $B=\{(a,c),(b,b),(c,a),(d,d)\}$, which are the rotation of 90^0 about the point o and the rotation of 180^0 about the line through b and d, respectively, generate the entire group. The eight elements

$$E = A^4 = B^2, A, A^2, A^3, B, AB, A^2B, A^3B$$

make up the group G of symmetries of the square. A students' task could be defining the symmetries described by the elements, different from E, A, B. We note only that the composite function $AB=\{(a,b),(b,a),(c,d),(d,c)\}$ is the rotation about the line through the midpoints of the sides joining a with b and c with d. The following Table 1 shows the operation table of the group:

$\bullet$	E	A	A^2	A^3	B	AB	A^2B	A^3B
E	E	A	A^2	A^3	B	AB	A^2B	A^3B
A	A	A^2	A^3	E	AB	A^2B	A^3B	B
A^2	A^2	A^3	E	A	A^2B	A^3B	B	AB
A^3	A^3	E	A	A^2	A^3B	B	AB	A^2B
B	B	A^3B	A^2B	AB	E	A^3	A^2	A
AB	AB	B	A^3B	A^2B	A	E	A^3	A^2
A^2B	A^2B	AB	B	A^3B	A^2	A	E	A^3
A^3B	A^3B	A^2B	AB	B	A^3	A^2	A	E

Table 1

From this Table 1 we see that the set $H=\{E, A^2\}$ is a subgroup, and checking the two products $AA^2 = A^2A = A^3$ and $BA^2 = A^2B$ we conclude that H is a normal subgroup of G (every element of G is composed of products of A and B).

We could give an example of a quotient group forming the group G/H whose elements are the cosets of H, namely $EH = H$, $AH=\{A,A^3\}$, $BH=\{B,A^2B\}$ and $ABH=\{AB,A^3B\}$. It is given on Table 2:

$\bullet$	EH	AH	BH	ABH
EH	EH	AH	BH	ABH
AH	AH	EH	ABH	BH .
BH	BH	ABH	EH	AH
ABH	ABH	BH	AH	EH

Table 2

Problems for students' work could be the following ones:
1/ find a subgroup of order 2 in G/H;
2/ find all maximal normal subgroups of G and their normal subgroups.

CONCLUSION

The review is only one approach for using more illustrations in teaching Group Theory. The audience and the lecturer are two parts of one process. The very important problem the university teaching to become more effective and the university education more sensible could not be easily solved but the long teachers' experience at the universities has to be taken into account in the process of solving it.

REFERENCES

[1] Deskins, W. E. ABSTRACT ALGEBRA, Dover Publications, New York, 1995.

[2] Durbin, J. R. MODERN ALGEBRA, AN INTRODUCTION, John Wiley & Sons, New York, 1992.

[3] Wolfram, St. *MATHEMATICA*, A System for Doing Mathematics by Computer, Addison- Wesley, Reading, 1993.

[4] Rashkova Ts. Lectures on Higher Mathematics I, Ruse Univ., 2009 (Bulgarian).

[5] Rashkova Ts. Algebra – theory and problems, Ruse Univ., 2011 (Bulgarian).

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ПРЕПОДАВАНЕ ТЕОРИЯ НА ГРУПИТЕ ЧРЕЗ ТРАНСФОРМАЦИИ

Цецка Рашкова

Русенски университет "Ангел Кънчев"

***Резюме**: Статията излага подход за по-добро разбиране основни идеи от Теория на групите, част от университетски курс по Алгебра в Русенския университет, чрез по-голяма застъпване на трансформациите и използване на софтуер за визуализация.*

***Ключови думи:** обща и специална линейни групи, елементи от краен ред, трансформации, визуализация.*

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