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AN APPLICATION OF THE SYMMETRIC GROUP IN COLOURING OBJECTS[1]

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Abstract: *The paper illustrates how basic ideas of Group Theory, a part of an university course in Algebra at the University of Ruse, could be better understood if we consider proper applications of interest for the students, namely properties of the symmetric group in regard to colouring objects.*

Keywords: *symmetric group, cyclic index of a group of permutations, colourings of objects with different colours.*

INTRODUCTION

The nowadays situation with the background of the first year students forces the teaching of the university courses to be in close connection with it. We think that better understanding of the material could be facilitated if more different kinds of examples prevail in the exposition and the knowledge from earlier taught courses is widely used. When a graduate student is working on his Bsc Degree thesis he could apply a broader view on the topic selected and he could both find connections being impossible earlier and come to a better understanding of the material earlier taught.

The paper gives an approach for achieving these goals concerning basic ideas of Group Theory. It shows how a graduate student in the speciality Pedagogics of Education in Mathematics and Informatics could interprete ideas from the university course in Algebra at the University of Ruse in her MsDegree thesis.

The review is based partially on [1,2,3,4] and we consider that the basic notions of Group Theory and the nature of the symmetric group S_n are well known.

Here we'll give the definitions for orbits and stabilizers and some of their properties.

Definition 1. Let G be the group of permutations of the set X. The relation $x \sim y \Leftrightarrow g(x) = y$ for $x, y \in X$ and $g \in G$ is reflexive, symmetric and transitive. The distinct equivalence classes of $\sim$ form a partition of X. The equivalent classes are called orbits of G in X.

The orbit Gx contains all elements of X, which are indistinguishable by the action of G, namely $Gx = \{y \in X : y = g(x), g \in G\}$.

Definition 2. For G, the group of permutations of the set X, we denote $G(x \to y) = \{g \in G : g(x) = y\}$.

The set $G_x = G(x \to x)$ is called stabilizer of X. It contains all the permutations in G which fix x. It is subgroup of G.

Theorem 1 [1, Theorem 14.2]. Let G be the group of permutations of the set X and suppose $h \in G(x \to y)$. Then $G(x \to y) = hG_x$ is the left coset of the subgroup G_x with respect to h.

There is a fundamental relationship between the size of an orbit Gx and the size of the stabilizer G_x.

[1]Partially supported by Grant 2016-FPNO-03 at the Angel Kanchev University of Ruse.

Theorem 2 [1, Theorem 14.3]. Let G be the group of permutations of the set X and $x \in X$. Then we have the equation $|Gx| \times |G_x| = |G|$.

We could illustrate the validity of Theorem 2 when G is the group of symmetries of a square, regarded as permutations of the corners labeled by 1,2,3,4 in clockwise direction starting from the left upper corner.

Another form of this group was given in [5].

The eight permutations of the considered group G are listed below:

Identity	**id**
Clockwise rotation through 90^0	**(1234)**
Clockwise rotation through 180^0	**(13)(24)**
Clockwise rotation through 270^0	**(1432)**
Reflection in diagonal (13)	**(24)**
Reflection in diagonal (24)	**(13)**
Reflection in perpendicular bisector of (12)	**(12)(34)**
Reflection in perpendicular bisector of (14)	**(14)(23)**

They form a group, which is a subgroup of S_4.

The orbit $G1$ of the corner 1 (say) is the whole set as G contains the permutations **id** (1 to 1), **(1234)** (1 to 2), **(13)(24)** (1 to 3) and **(1432)** (1 to 4). Thus $|G1| = 4$. The stabilizer of 1 is $G_1 = \{id, (24)\}$ and so $|G1| \times |G_1| = 4 \times 2 = 8$, as expected, since there are eight symmetries in all.

Given any group G of permutations of a set X we define, for each g in G, a set $F(g) = \{x \in X \mid g(x) = x\}$. Thus $F(g)$ is the set of objects fixed by g.

Theorem 3 [1, Theorem 14.4] . The number of orbits of G on X is

$$\frac{1}{|G|} \sum_{g \in G} |F(g)|.$$

CYCLE INDEX OF THE GROUP OF PERMUTATIONS

The most important tool in studying the number of distinguishable due to the symmetry colourings of an object is the compact notation giving information about the cycle structures of permutations in a group.

Given any group G of permutations of the set $X = \{1,2,\dots,n\}$ for each g in G we define the type of g as the corresponding partition $[1^{\alpha_1} 2^{\alpha_2} .. n^{\alpha_n}]$ of length n, meaning that g contains α_1 cycles of length 1, α_2 cycles of length 2 ,..., α_n cycles of length n.

We have $\alpha_1 + 2\alpha_2 + \dots + n\alpha_n = n$.

We associate with g an expression

$$\varsigma_g(x_1, x_2, \dots, x_n) = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n},$$

where the x_i $(1 \le i \le n)$ are simply formal symbols like the x in a polynomial.

The formal sum of the ς_g, taken over all g in G, is a polynomial in x_1 , $x_2, \dots, x_n$. Dividing by $|G|$ we obtain the **cycle index** of the group of permutations:

$$\varsigma_G(x_1, x_2, \ldots, x_n) = \frac{1}{|G|}\sum_{g \in G} \varsigma_g(x_1, x_2, \ldots, x_n) = \frac{1}{|G|}\sum_{g \in G} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}\ .$$

If G is the group of symmetries of a square, regarded as permutations of the corners, we give the expressions ς_g in the following table:

Thus the cycle index of the group of the square, as considered above, is

$$\frac{1}{8}(x_1^4 + 2x_1^2 x_2 + 3x_2^2 + 2x_4), \qquad (1)$$

where the terms corresponding to permutations of the same type are collected.

THE NUMBER OF NONEQUIVALENT COLOURINGS

Here we consider the general problem of finding the number of distinguishable colourings when a group of permutations is involved.

An example is the problem of black-and-white colourings of the corners of a square. Since there are two colours and four corners there are basically $2^4 = 16$ possibilities. But when we take account of the symmetry of the square we see that some of the possibilities are essentially the same.

For example the two colourings on Figure 1 are one and the same (the first one is the same as the second one after rotation through 180^0).

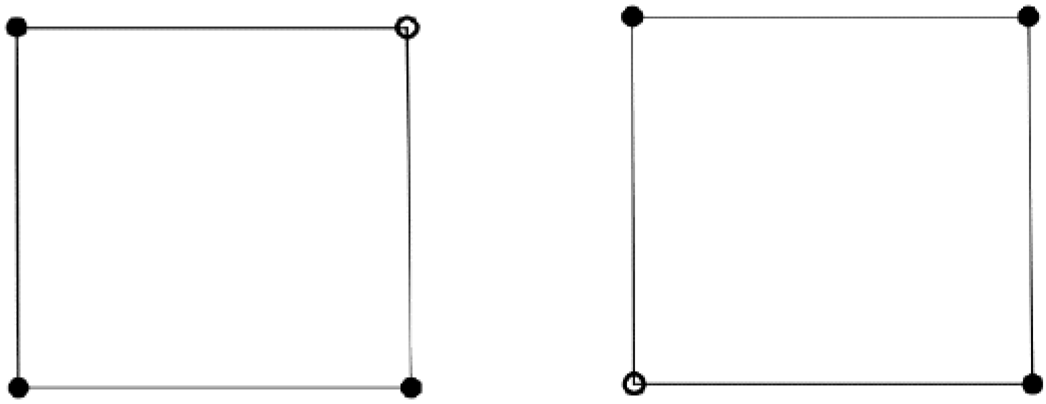

Fig. 1

Thus we regard two colourings as being indistinguishable if one is transformed into the other by a symmetry of the square.

In this case it is easy to find by trial and error that there just six of them, as shown in Figure 2.

In general, suppose we have a group G of permutations of an n-set X, and to each element of X can be assighned one of r colours. If we denote the set of the colours by K, then a **colouring** is simply a function ω from X to K. There are r^n colourings in all, denoted the set of them by Ω.

Each permutation g in G induces a permutation $\overline{g}$ of Ω in the following way:

Given a colouring ω, we define $\overline{g}(\omega)$ to be the colouring in which the colour assighend to x is the colour ω assigns to $g(x)$, i.e.

$$(\overline{g}(\omega))(x) = \omega(g(x)).$$

The function taking g to $\overline{g}$ is a representation of G as a group $\overline{G}$ of permutations of Ω.Two colourings are indistinguishable if one of them can be transformed into the other by some permutation $\overline{g}$, i.e. if they belong to the same orbit of $\overline{G}$ on Ω. Thus the number of distinguishable colourings is just the number of orbits in the action of $\overline{G}$ on Ω.

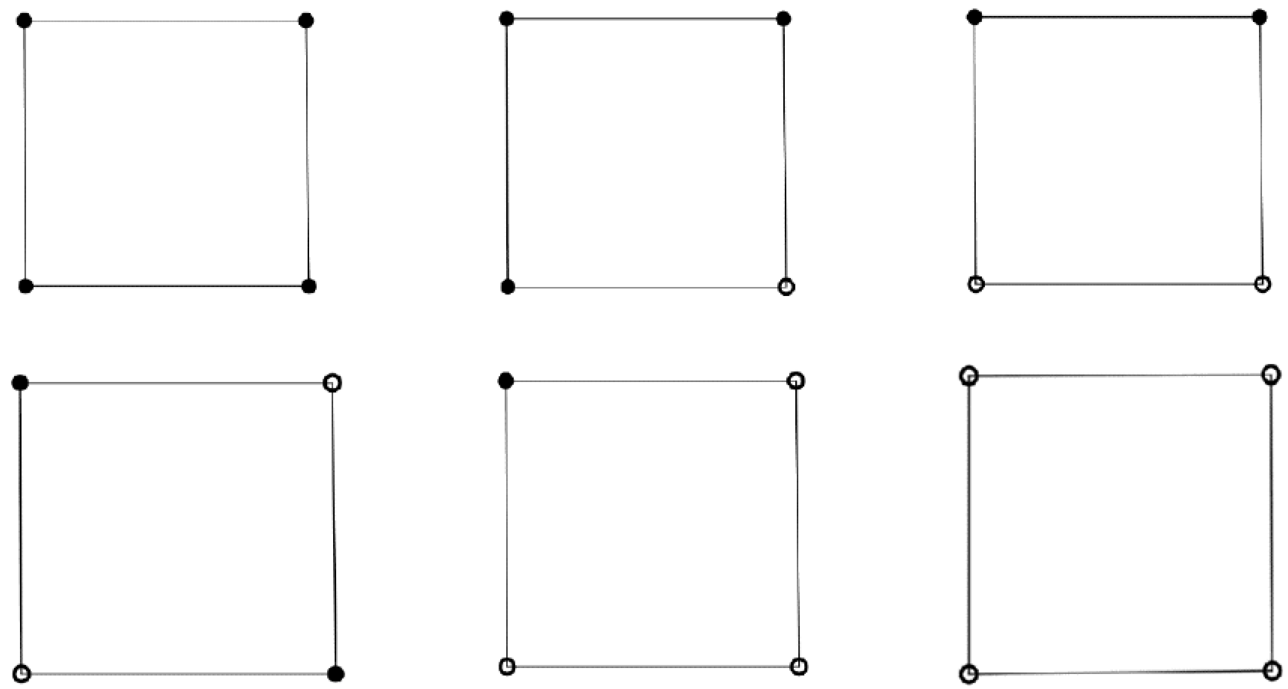

Fig. 2

Proposition 1. The image group $\overline{G}$ is isomorphic to G.

Proof: Suppose that $\overline{g_1} = \overline{g_2}$, i.e. $\omega(g_1(x)) = \omega(g_2(x))$, $\omega \in \Omega, x \in X$. Since this equation is true for all ω, it is true in particular for the colouring which assigns a specified colour to $g_1(x)$ and another colour to every other member of X. Thus $g_1(x) = g_2(x)$ and it is true for each $x \in X$, i.e. $g_1 = g_2$.

Thus applying Theorem 3 we get

Theorem 4 [1, Theorem 20.4]. If G is a group of permutaions of X and $\varsigma_G(x_1, x_2, ..., x_n)$ is its cycle index, then the number of inequivalent colourings of X with r colours available is $\varsigma_G(r, r, ..., r)$.

Thus the problem of finding the number of inequivalent colourings when r colours are available can be reduced to the problem of calculating the cycle index. When the cycle index is known, we have only to replace each of $x_1, x_2, ..., x_n$ by r in order to get the result.

Example 1. The number of inequivalent colourings with two colours of the corners of a square is 6.

Solution: The number of inequivalent colourings of the corners of a square is $\frac{1}{8}(r^4 + 2r^3 + 3r^2 + 2r)$, obtained by putting $x_1 = x_2 = x_3 = x_4 = r$ in (1).

When $r = 2$ we obtain $\frac{1}{8}(2^4 + 2.2^3 + 3.2^2 + 2.2) = 6$, in agreement with Figure 2.

Example 2. Participants from an university at a sport event are distinguished not, as it is usual, by wearing of numbers, but by rectangular badges made from four pieces of coloured material, as in Figure 3.

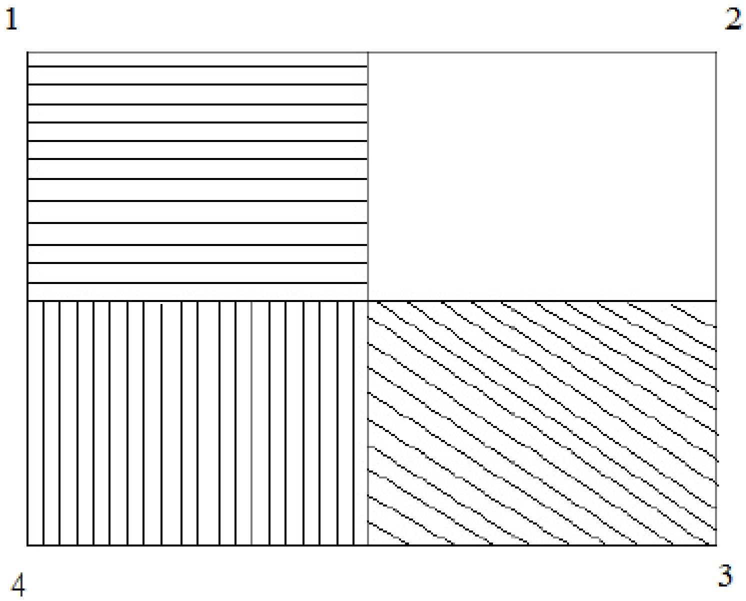

Fig. 3

An usual event however for the participants is to pin on their badges upside-down, or back-to-front, or both. If the expected participants are160, what is the smallest number of coloured materials required to make the badges?

Solution: The segments of the badge can be identified with the corners of the rectangle, which we shall label 1,2,3,4 in clockwise order with 1 at the top left.

The relevant group of permutations consists of the permutations:

Correct position	**id**
Upside-down	**(13)(24)**
Back-to-front	**(12)(34)**
Both (u-d, b-f)	**(14)(23)**

Thus the cycle index for the group of the rectangle is $\frac{1}{4}(x_1^4 + 3x_2^2)$, and the number of badges with r colours available is $\frac{1}{4}(r^4 + 3r^2)$.

The smallest integer r for which $\frac{1}{4}(r^4 + 3r^2) \geq 160$ is 5, so this is the number of colours required.

Considering the groups of the symmetries of other plane or surface objects we could form a greater collection of interesting problems connected with colouring of objects.

CONCLUSION

The review is only one example how in additional way (working with the students on projects or theses) we could encourage students to show bigger interest in topics from disciplines in the regular program traditionally being of small interest due to different reasons.

Colouring of objects is of interest not only for students. It could have future applications in teachers' everyday work with pupils at school as well. Thus what is given here could be continued further on and is of evident significance for future teachers in mathematics.

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ПРИЛОЖЕНИЕ НА СИМЕТРИЧНАТА ГРУПА ПРИ ОЦВЕТЯВАНЕ НА ОБЕКТИ

Цецка Рашкова, Надежда Данова

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***Резюме**: Статията илюстрира как основни идеи от Теория на групите, част от университетски курс по Алгебра в Русенския университет, могат да бъдат разбрани по-добре от студентите при разглеждане подходящи приложения от интерес за обучаемите, а именно някои свойства на симетричната група, приложими при оцветяване на обекти.*

***Ключови думи:** симетрична група, цикличен индекс на група от пермутации, оцветяване на обекти с различни цветове.*

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