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# PI-PROPERTIES OF A FOURTH ORDER MATRIX SUPERALGEBRA WITH INVOLUTION

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***Abstract***: *A special fourth order matrix superalgebra $\mathrm{A}$ over the Grassmann algebra is defined, an involution $\varphi$ is introduced in it and its symmetric and skew symmetric due to the involution elements are described. A set of $\varphi$-polynomial identities of degree $\leq 5$ for the considered algebra is given. All two variables' $\varphi$-identities of degrees 3, 4 are described. A description is given as well of the monomes of arbitrary degree in $\mathrm{A}$ with the respective limitations on the number, type and the degree of the variables in the monomes.*

***Keywords***: *Grassmann algebra, matrix algebras with Grassmann entries, involution $\varphi$, symmetric and skew symmetric variables, $\varphi$-polynomial identities.*

## INTRODUCTION

We consider a fourth order matrix superalgebra $\mathrm{A}$ with involution $\varphi$ over the infinite dimensional Grassmann algebra $E$ and describe some of its $\varphi$-polynomial identities. A similar question is discussed in [2], where the algebra $M_{1,1}(E)$ endowed with the involution $\varphi$ induced by the transposition superinvolution of the superalgebra $M_{1,1}(F)$ of $2\times 2$-matrices over the field $F$ is considered. In [2] the authors describe a finite set generating the ideal of the $\varphi$-identities of $M_{1,1}(E)$. In our case we still could not give the generators of the $T_\varphi$-ideal of the $\varphi$-identities of $\mathrm{A}$ but we give a set of $\varphi$-polynomial identities of degree $\leq 5$ for the algebra $\mathrm{A}$.

We recall the definition of the infinite dimensional Grassmann algebra $E$ as

$$E = E(V) = K\langle e_1, e_2, \ldots \mid e_i e_j + e_j e_i = 0 \quad i, j = 1,2,\ldots\rangle,$$

where the field $K$ has characteristic zero. The elements $e_i$ are called generators of $E$, while the elements $e_{i_1} e_{i_2} \ldots e_{i_k}$ for $1 \leq i_1 \leq i_2 \ldots \leq i_k$ are called basic monomials of $E$. The element $1$ is a generator as well.

The algebra $E$ is in the mainstream of recent research in PI-theory. Its importance is connected with the structure theory for the $T$- ideals of identities of associative algebras developed by Kemer who proved that any $T$- prime $T$- ideal can be obtained as the $T$-ideal of identities of one of the following algebras: $M_n(K)$, $M_n(E)$ and $M_{n,u}(E)$, the latter being the algebra of $n\times n$ supermatrices over $E = E_0 \oplus E_1$ with two $E_0$ blocks (with entries of even degree) of sizes $u\times u$ and $(n-u)\times(n-u)$ and with two $E_1$ blocks (with entries of odd degree) of sizes $u\times(n-u)$ and $(n-u)\times u$.

Another reason for the Grassmann algebra to be one of the fundamental structures in PI-theory is the fact that it generates a minimal variety of exponential growth [4].

The importance of considering the matrix algebra with Grassmann entries $M_n(E)$ is confirmed by the following statement as the trivial isomorphism $E \otimes M_n(K) \cong M_n(E)$ holds:

**Proposition 1 [3, Corollary 8.2.4].** *For every PI-algebra $R$ there exists a positive $n$ such that $T(R) \supseteq T(M_n(E))$, i.e. $R$ satisfies all polynomial identities of the $n \times n$ matrix algebra $M_n(E)$ with entries from the Grassmann algebra.*

Many of the PI-properties of $E$ and $M_n(E)$ could be found in [4, 1]. Here we formulate some of them:

**Proposition 2 [4, Corollary, p. 437].** *The $T$-ideal $Id(E)$ is generated by the identity $[x_1, x_2, x_3] = [x_1, x_2]x_3 - x_3[x_1, x_2] = 0$ (called the Grassmann identity).*

**Proposition 3 [1, Lemma 6.1].** *The algebra $E$ satisfies $S_n(x_1, \ldots, x_n)^k = 0$ for all $n, k \geq 2$ and $S_n(x_1, \ldots, x_n) = \sum_{\sigma \in Sym(n)} (-1)^\sigma x_{\sigma(1)} \ldots x_{\sigma(n)}$* being the standard identity.

**Proposition 4 [1, Corollary 6.6].** *The algebra $M_n(E)$ does not satisfy the identity $S_m^n(X_1, ..., X_m) = 0$* for any $m$.

## THE ALGEBRA AND THE INVOLUTION

Let $\varphi$ be an involution in the algebra $\mathrm{A}$ ($\varphi(ab) = \varphi(b)\varphi(a)$ for all $a, b \in \mathrm{A}$) . By $\mathrm{A}^-$ we denote the skew symmetric due to the involution elements $z_1, ..., z_m, ..$ of $\mathrm{A}$ ($\varphi(z) = -z$) and by $\mathrm{A}^+$ we denote the symmetric due to the involution elements $y_1, ..., y_j, ...$ ($\varphi(y) = y$).

**Definition 1.** *Let $f = f(x_1, \ldots, x_m) \in K\langle x_1, ..., x_n \rangle$, the free associative algebra on $n$ generators over $K$. We say that $f$ is a $\varphi$-identity in skew variables for the algebra $\mathrm{A}$ over $K$ if $f = f(z_1, \ldots, z_m) = 0$ for all $z_1, ..., z_m \in \mathrm{A}^-$. Accordingly $f$ is a $\varphi$-identity in symmetric variables for the algebra $\mathrm{A}$ over $K$ if $f = f(y_1, \ldots, y_m) = 0$ for all $y_1, ..., y_m \in \mathrm{A}^+$.*

*The polynomial $f = f(x_1, \ldots, x_m)$ for $m \geq 2$ is a $\varphi$-identity for $\mathrm{A}$ if $f = f(z_1, \ldots, z_k, y_{k+1}, ..., y_m) = 0$ where $1 \leq k \leq m-1$, $z_1, ..., z_k \in \mathrm{A}^-$ and $y_{k+1}, ..., y_m \in \mathrm{A}^+$.*

We consider the algebra $\mathrm{A}$ of the matrices over $E$ of type $\begin{pmatrix} y_1 & 0 & z_1 & 0 \\ 0 & y_2 & 0 & z_2 \\ z_3 & 0 & y_3 & 0 \\ 0 & z_4 & 0 & y_4 \end{pmatrix}$, for which $\begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix}$ and $\begin{pmatrix} y_3 & 0 \\ 0 & y_4 \end{pmatrix}$ have even entries, while the matrices $\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}$ and $\begin{pmatrix} z_3 & 0 \\ 0 & z_4 \end{pmatrix}$ have odd entries.

We recall that $E = E_0 \oplus E_1$, where the elements of $E_0$ have even degree, while the elements of $E_1$ have odd degree.

In the algebra $\mathrm{A}$ we introduce the following mapping $\varphi$:

$$\begin{pmatrix} y_1 & 0 & z_1 & 0 \\ 0 & y_2 & 0 & z_2 \\ z_3 & 0 & y_3 & 0 \\ 0 & z_4 & 0 & y_4 \end{pmatrix}^{\varphi} = \begin{pmatrix} y_3 & 0 & z_1 & 0 \\ 0 & y_4 & 0 & z_2 \\ -z_3 & 0 & y_1 & 0 \\ 0 & -z_4 & 0 & y_2 \end{pmatrix}.$$

**Proposition 5.** *The mapping $\varphi$ is an involution in the algebra $\mathrm{A}$.*

**Proof:** For any matrices $A$ as above and $B = \begin{pmatrix} y_5 & 0 & z_5 & 0 \\ 0 & y_6 & 0 & z_6 \\ z_7 & 0 & y_7 & 0 \\ 0 & z_8 & 0 & y_8 \end{pmatrix}$ we form

$$(AB)^{\varphi} = \begin{pmatrix} z_3z_5 + y_3y_7 & 0 & y_1z_5 + z_1y_7 & 0 \\ 0 & z_4z_6 + y_4y_8 & 0 & y_2z_6 + z_2z_8 \\ -z_3y_5 - y_3z_7 & 0 & y_1y_5 + z_1z_7 & 0 \\ 0 & -z_4y_6 - y_4z_8 & 0 & y_2y_6 + z_2z_8 \end{pmatrix} \text{ and}$$

$$B^{\varphi}A^{\varphi} = \begin{pmatrix} y_7y_3 - z_5z_3 & 0 & y_7z_1 + z_5y_1 & 0 \\ 0 & y_8y_4 - z_6z_4 & 0 & y_8z_2 + z_6y_2 \\ -z_7y_3 - y_5z_3 & 0 & -z_7z_1 + y_5y_1 & 0 \\ 0 & -z_8y_4 - y_6z_4 & 0 & -z_8z_2 + y_6y_2 \end{pmatrix}.$$

As all $y$'s are even and all $z$'s are odd we get that $(AB)^{\varphi} = B^{\varphi}A^{\varphi}$.

We give the form of the symmetric due to the involution variables $Y_i$ of the algebra $\mathrm{A}$ and of the skew symmetric ones $Z_i$, namely

$$Y_i = \begin{pmatrix} \alpha_{i1} & 0 & \beta_{i1} & 0 \\ 0 & \alpha_{i2} & 0 & \beta_{i2} \\ 0 & 0 & \alpha_{i1} & 0 \\ 0 & 0 & 0 & \alpha_{i2} \end{pmatrix}; \; Z_i = \begin{pmatrix} a_{i1} & 0 & 0 & 0 \\ 0 & a_{i2} & 0 & 0 \\ c_{i1} & 0 & -a_{i1} & 0 \\ 0 & c_{i2} & 0 & -a_{i2} \end{pmatrix} \qquad (1)$$

for $\alpha_{ij}, a_{ij} \in E_0$, $\beta_{ij}, c_{ij} \in E_1$.

We'll rely on these presentations in the calculations made below in proving the statements in the paper.

The investigation of both the symmetric and the skew symmetric elements of an algebra $\mathrm{A}$ is an important problem as the set $\mathrm{A}^+$ of the symmetric elements form a Jordan algebra due to the operation $a \circ b = ab + ba$ and the set $\mathrm{A}^-$ of its skew symmetric elements form a Lie algebra due to the operation $[a,b] = ab - ba$.

## THE PI-STRUCTURE OF THE ALGEBRA $\mathrm{A}$

**Proposition 6.** *The algebra* $\mathrm{A}$ *satisfies the following set of one* $\varphi$*-polynomial identity in symmetric elements and two more in skew symmetric ones, namely:*

$$(a) \qquad [Y_1, Y_2] = 0;$$

$$(b) \quad Z_1 Z_2 Z_3 - Z_3 Z_2 Z_1 = 0;$$

$$(c) \quad [Z_1, Z_2][Z_3, Z_4] = 0.$$

**Proof:** By direct calculations using the above given presentation (1) of the corresponding elements.

It is easy to get that

$$[Z_i, Z_j] = 2(c_{i1}a_{j1} - a_{i1}c_{j1})e_{31} + 2(c_{i2}a_{j2} - a_{i2}c_{j2})e_{42};$$

$$Z_i \circ Z_j = 2a_{i1}a_{j1}(e_{11} + e_{33}) + 2a_{i2}a_{j2}(e_{22} + e_{44});$$

$$Y_i \circ Y_j = 2\alpha_{i1}\alpha_{j1}(e_{11} + e_{33}) + 2\alpha_{i2}\alpha_{j2}(e_{22} + e_{44}) + 2(\alpha_{i1}\beta_{j1} + \alpha_{j1}\beta_{i1})e_{13} + 2(\alpha_{i2}\beta_{j2} + \alpha_{j2}\beta_{i2})e_{24};$$

$$[Y_i, Z_j] = \beta_{i1}c_{j1}(e_{11} + e_{33}) + \beta_{i2}c_{j2}(e_{22} + e_{44}) - 2\beta_{i1}a_{j1}e_{13} - 2\beta_{i2}a_{j2}e_{24}.$$

The following two 3-degree presentations are valid as well:

$$[Y_i, Z_j, Z_k] = 2c_{k1}\beta_{i1}a_{j1}(e_{11} + e_{33}) + 2c_{k2}\beta_{i2}a_{j2}(e_{22} + e_{44}) + 4\beta_{i1}a_{j1}a_{k1}e_{13} + 4\beta_{i2}a_{j2}a_{k2}e_{24};$$

$$[Z_i, Z_j, Y_k] = 2(c_{i1}a_{j1} - a_{i1}c_{j1})\beta_{k1}(e_{11} + e_{33}) + 2(c_{i2}a_{j2} - a_{i2}c_{j2})\beta_{k2}(e_{22} + e_{44}).$$

The above relations and the nature of the corresponding odd or even entries lead to the validity of the following

**Proposition 7.** *The following* $\varphi$ *– identities hold:*

$$[Y_i, Z_j, Y_k] = 0; \qquad (2)$$

$$[Z_i, Z_j]Y_k[Z_m, Z_n] + [Z_m, Z_n]Y_k[Z_i, Z_j] = 0; \qquad (3)$$

$$[Y_i, Z_i][Y_j, Z_j] + [Y_i, Z_j][Y_j, Z_i] = 0; \qquad (4)$$

$$[Z_i, Z_j, Y_k, Z_l] = 0; \qquad (5)$$

$$[Z_i, Z_j, Y_k, Y_l] = 0. \qquad (6)$$

**LOW DEGREES' $\varphi$- IDENTITIES IN TWO VARIABLES**

It is seen (from the relations after the proof of Proposition 6) that $Z \circ Z$ is a diagonal matrix. Thus we get the $\varphi$-identity

$$[Z^2, Y] = 0. \tag{7}$$

For $Y_i = [Z, Y]$ and $Y_j = Y$ in Proposition 6 (a) we see that

$$[Z, Y, Y] = 0. \tag{8}$$

**Proposition 8.** *The identities* (7) *and* (8) *are the only* 3 *degree* $\varphi$ *– identities in two variables in the algebra* $\mathrm{A}$.

**Proof:** We give the considerations made ommitting the details of the direct calculations.

Using (7) we get $YZ^2 = Z^2Y$. Identity (8) gives $Y^2Z + ZY^2 = 2YZY$. Thus there are only two possibilities for such $\varphi$ – identities, namely

$$\alpha ZYZ + \beta Z^2Y = 0,$$
$$\gamma ZY^2 + \delta Y^2Z = 0$$

for $\alpha, \beta, \gamma, \delta \in K$.

Using the above given presentation (1) of a symmetric element $Y$ and a skew symmetric element $Z$ of the considered algebra we see that the two identities are possible only in the trivial case, for zero values of the elements of $K$.

**Proposition 9.** *The only* $\varphi$ *– identities of degree* 4 *in two variables in the algebra* $\mathrm{A}$ *are the consequences of* (7) *and* (8).

**Proof:** We follow the same scheme as above considering 3 cases:

1) **The degrees of $Y$ and $Z$ are equal.** A consequence of (7) is $Y^2Z^2 = YZ^2Y = Z^2Y^2$ while (8) is equivalent to $ZY^2 + Y^2Z = 2YZY$. Thus we get $ZYZY = YZYZ$. The only possible identity could be of type $f_1(Y, Z) = \alpha_1 YZ^2Y + \beta_1 YZYZ = 0$. Writing directly the two summands in it we see that there is no a non-zero solution for the coefficients $\alpha_1$ and $\beta_1$.

2) **The degree of $Y$ is one**. From (7) we get $YZ^3 = Z^2YZ$ and $Z^3Y = ZYZ^2$. Thus the only possible identity could be of type $f_2(Y, Z) = \alpha_2 YZ^3 + \beta_2 Z^3Y = 0$. The detailed explicit form of it leads again to zero values of the two coefficients in it.

3) **The degree of $Y$ is three.** Due to identity (8) we have

$$Y^2Z + ZY^2 = 2YZY$$
$$YZY^2 = \frac{1}{3}(Y^3Z + 2ZY^3).$$
$$Y^2ZY = \frac{1}{3}(ZY^3 + 2Y^3Z)$$

The possible identity could be only of type $f_3(Y, Z) = \alpha_3 Y^3Z + \beta_3 ZY^3 = 0$. Again its explicit form gives that $\alpha_3 = \beta_3 = 0$.

**Corrolary 1.** *The* $\varphi$ *– identities in two variables of degree* 5 *in the algebra* $\mathrm{A}$ *could be only of the following three types*:

$$\alpha_1 Y^2 Z^3 + \beta_1 Z^3 Y^2 + \gamma_1 Z^2 YZY + \delta_1 ZYZYZ = 0;$$

$$\alpha_2 Y^3 Z^2 + \beta_2 YZY^2 Z + \gamma_2 ZY^2 ZY + \delta_2 Y^2 ZYZ + \rho YZYZY = 0;$$

$$\alpha_3 Y^4 Z + \beta_3 ZY^4 = 0$$

*for suitable values of the coefficients* $\alpha_i, \beta_i (i = 1,2,3); \gamma_j, \delta_j (j = 1,2); \rho \in K$.

The monomes of degree 4 show the type of the corresponding monomes of arbitrary degree $n$ as well:

**Proposition 10.** *The monomials of the algebra* $\mathrm{A}$ *in one symmetric variable* $Y$ *of degree not greater than* 2 *and in one skew symmetric variable* $Z$ *are of the following types*:

(a) *For degree* 1 *of* $Y$ *they are* $YZ^{n-1}$ *and* $Z^{n-1}Y$;

(b) *For degree* 2 *of* $Y$ *they are* $Y^2 Z^{n-2}$, $Z^{n-3}Y^2 Z$ *and* $Z^{n-4}YZYZ$;

**Proposition 11.** *The monomials of degree* $n$ *of the algebra* $\mathrm{A}$ *in one symmetric variable* $Y$ *of degree greater than* 2 *and in one skew symmetric variable* $Z$ *are of the following types*:

$Y^s Z^{n-s}$ *if* $n-s$ *is even or* $Y^s Z^{n-s}$ *and* $Z^{n-s}Y^s$ *if* $n-s$ *is odd*;

$Z^{n-s-l}Y^{k_1}ZY^{k_2}...ZY^{k_l}ZY^{s-p}$ *for* $p = k_1 + k_2 + ... + k_l$, $l \le p \le s$, $l = 1,...,n-s$.

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# PI-СВОЙСТВА НА МАТРИЧНА СУПЕРАЛГЕБРА ОТ ЧЕТВЪРТИ РЕД С ИНВОЛЮЦИЯ

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***Резюме***: *Дефинирана е матрична супералгебра $\mathrm{A}$ от четвърти ред над безкрайномерната Грасманова алгебра, въведена е инволюция $\varphi$ в нея и са описани симетричните ѝ и антисиметрични относно инволюцията елементи. Посочено е множество от $\varphi$-полиномни тъждества от степен $\leq 5$ за разглежданата алгебра. Описани са всички $\varphi$-тъждества на две променливи от степени 3 и 4. Направено е описание и на мономите от произволна степен в $\mathrm{A}$ при съответни ограничения върху броя, типа и степента на променливите в тях.*

***Ключови думи*:** *Грасманова алгебра, матрична алгебра с грасманови елементи, инволюция $\varphi$, симетрични и антисиметрични променливи, $\varphi$-полиномни тъждества.*

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