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ON THE SOLVABILITY OF A SINGULAR SECOND ORDER INITIAL VALUE PROBLEMS

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Abstract: *Using the Topological transversality theorem and the barrier strips technique, we study the solvability of second order initial value problems singular at the initial value of the independent variable. The obtained result guarantees the existence of at least one monotone solution which does not change its sign.*
Keywords: *Initial value problems, ordinary differential equations, singularity, existence, monotone and positive solutions, barrier technique.*

INTRODUCTION

Here we consider the initial value problem (IVP)

$$\begin{cases} x'' = f(t, x, x'), \\ x(0) = A,\ x'(0) = B, A, B \geq 0, \end{cases} \tag{1.1}$$

where the scalar function $f(t,x,p)$ is defined for $(t,x,p) \in D_t \times D_x \times D_p$, $D_t,\ D_x,\ D_p \subseteq \mathbf{R}$, but may there are sets $\mathrm{X} \subseteq D_x$ and $\mathrm{P} \subseteq D_p$ such that f to be unbounded as $t \to 0^+$ and $(x,p) \in \mathrm{X} \times \mathrm{P}$.

Various IVPs, nonsingular and singular, have been studied by A. Aslanov [1], Agarwal and O'Regan [2,3], L. Bobisud and D. O'Regan [4], L. Bobisud and Y. Lee [5], A. Cabada et al. [6-8], J. Cid [9], H. Maagli and S. Masmoudi [13], I. Rachůnková and J. Tomeček [14-16], G. Yang [17,18] and Z. Zhao [19]. In [5], for example, the IVP

$$(\varphi(t)x'(t))' = q(t) f(t, x(t), \varphi(t)x'(t)),$$
$$x(0) = A > 0,\ \lim_{t \to 0^+} \varphi(t)x'(t) = B,$$

has been considered, where $1/\varphi, q \in L^1(0,T)$ are suitable functions. It is assumed that there are positive constants T, M and a continuous, non-decreasing function $g(x) \geq 0,\ x \in [0,\infty)$, such that either $g \equiv 0$ or

$$\lim_{x \to \infty} \frac{x}{g(x)} > \int_0^T \frac{1}{\varphi(s)} \int_0^s q(t)\,dt\,ds$$

and

$$f(t,x,p) \leq g(x) \text{ on } [0,T]x[0,\infty) \times \mathbb{R} \text{ with } inf_{[0,T]\times[0,M]\times\mathbb{R}} f(t,x,p) > -\infty$$

Under these assumptions the authors have shown that the IVP has $C[0,T] \cap C^2(0,T)$-solutions, and similar assumptions guarantee a monotonically increasing solution.

IVPs of the form (1.1) have been studied in P. Kelevedjiev and N. Popivanov [10], P. Kelevedjiev et al. [11] and P. Kelevedjiev [12]. Moreover, in [10] and [11], the nonlinearity f may be unbounded at $x = A$ and/or $p = B$, and in [12] – at $t = 0$. Barrier type conditions are used in these works. In [10] and [11], they are imposed together with the assumption that there is a constant $k < 0$ such that

$$f(t,x,p) \leq k \tag{1.2}$$

on a suitable bounded subset of the domain of f. The following condition is imposed in [12].

S. There are constants $T > 0, m_1, \bar{m}_1, M_1, \bar{M}_1$ and a sufficiently small $\tau > 0$ such that

$$\bar{M}_1-\tau \ge M_1 \ge B \ge m_1 \ge \bar{m}_1+\tau,\ (0,T]\subseteq D_t,\ [\bar{m}_1,\bar{M}_1]\subseteq D_p,$$

$$[\tilde{m}_0-\tau,\tilde{M}_0+\tau]\subseteq D_x,\ f(t,x,p)\in C\big((0,T]\times[\tilde{m}_0-\tau,\tilde{M}_0+\tau]\times[m_1-\tau,M_1+\tau]\big), \qquad (1.3)$$

where $\tilde{M}_0=\max\{|m_1|,|M_1|\}T+A,$ and $\tilde{m}_0=-\tilde{M}_0,$

$$f(t,x,p)\le 0 \text{ for } (t,x,p)\in(0,T]\times D_x\times[M_1,\bar{M}_1],$$

$$f(t,x,p)\ge 0 \text{ for } (t,x,p)\in(0,T]\times D_{\tilde{M}_0}\times[\bar{m}_1,m_1],$$

where $D_{\tilde{M}_0}=D_x\cap(-\infty,\tilde{M}_0].$

Under this assumption we have:

Theorem 1.1. ([12], Theorem 2.1) Let **S** hold. Then singular IVP (1.1) has at least one solution in $C^1[0,T]\cap C^2(0,T]$ such that

$$m_1t+A\le x(t)\le M_1t+A \text{ and } m_1\le x'(t)\le M_1 \text{ for } t\in[0,T].$$

It turned out, however, the requirements (1.3) may be weakened when $B\ge 0$. This is possible since this case allows us to choose non-negative m_1. The present paper is devoted to this possibility. So, we suppose here that:

S₁. The assumption **S** holds for $m_1\ge 0$ with (1.3) replaced by

$$[A-\tau,\tilde{M}_0+\tau]\subseteq D_x,\ f(t,x,p)\in C((0,T]\times[A-\tau,M_0+\tau]\times[m_1-\tau,M_1+\tau]),$$

where $\tilde{M}_0=M_1T+A,$ and $\tilde{m}_0=-\tilde{M}_0.$

Our main result is the following:

Theorem 1.2. Let $A>0\,(A=0)$, $B\ge 0$ and let **S₁** hold. Then singular IVP (1.1) has at least one positive (non-negative) non-decreasing solution in $C^1[0,T]\cap C^2(0,T]$ with the properties

$$m_1t+A\le x(t)\le M_1t+A \text{ for } t\in[0,T],$$

$$m_1\le x'(t)\le M_1 \text{ for } t\in[0,T].$$

By an example, we demonstrate the advantage of Theorem 1.2 before Theorem 1.1.

AUXILIARY RESULTS

To prove our existence theorem we need two results [10] for the nonsingular IVP

$$\begin{cases} x''=f(t,x,x'), \\ x(a)=A,\ x'(a)=B,\ A,B\ge 0, \end{cases} \qquad (2.1)$$

where $: Dt\times Dx\times Dp\to\mathbb{R}$, $Dt\times Dx\times Dp\subseteq\mathbb{R}$. They rely on the following barrier condition.

R. ([10]) There exists constants $T>a,\ m_1,\bar{m}_1,M_1,\bar{M}_1$ and a sufficiently small $\tau>0$ such that

$$m_1\ge 0,\ \bar{M}_1-\tau\ge M_1\ge B\ge m_1\ge\bar{m}_1+\tau,\ [a,T]\subseteq D_t,\ [A-\tau,M_0+\tau]\subseteq D_x,\ [\bar{m}_1,\bar{M}_1]\subseteq D_p,$$

where $M_0=M_1(T-a)+A,$

$$f(t,x,p)\in C\big([a,T]\times[A-\tau,M_0+\tau]\times[m_1-\tau,M_1+\tau]\big),$$

$$f(t,x,p)\le 0 \text{ for } (t,x,p)\in[a,T]\times D_x\times[M_1,\bar{M}_1],$$

$$f(t,x,p)\ge 0 \text{ for } (t,x,p)\in[a,T]\times D_{M_0}\times[\bar{m}_1,m_1],$$

where $D_{M_0}=D_x\cap(-\infty,M_0].$

Lemma 2.1. ([10], Lemma 3.1) Let **R** hold. Then each solution $x\in C^2[a,T]$ to (2.1) satisfies the bounds

$$A\le x(t)\le M_0,\ m_1\le x'(t)\le M_1,\ m_2\le x''(t)\le M_2 \text{ for } t\in[a,T],$$

where

$$m_2 = \min\{f(t,x,p) : (t,x,p) \in [a,T]\times[A,M_0]\times[m_1,M_1]\},$$
$$M_2 = \max\{f(t,x,p) : (t,x,p) \in [a,T]\times[A,M_0]\times[m_1,M_1]\}.$$

Theorem 2.1. ([10], Theorem 3.4) Let $A>0\,(A=0)$, $B\ge 0$ and let **R** hold. Then problem (2.1) has at least one positive (non-negative) non-decreasing solution in $C^2[a,T]$.

PROOF OF THE MAIN RESULT

Proof. We consider the following family of nonsingular IVPs

$$\begin{cases} x'' = f(t,x,x'), \\ x(n^{-1}) = A,\ x'(n^{-1}) = B, \end{cases} \tag{3.1}$$

where $n \in N_T = \{n \in \mathbb{N} : n^{-1} < T\}$

Firstly, we check easily that for each of regular problems (3.1) hypothesis $\mathbf{S_1}$ becomes to **R** with $a = n^{-1}$ and $M_0 = M_1(T - n^{-1}) + A$. Thus, according to Theorem 2.1, the corresponding problem of (3.1) has a solution $x_n \in C^2[n^{-1},T]$ for each $n \in N_T$. For each of these solutions Lemma 2.1 provides the bounds

$$A \le x_n(t) \le M_0 < \tilde{M}_0,\ \ t \in [n^{-1},T], \tag{3.2}$$

and

$$m_1 \le x_n'(t) \le M_1,\ \ t \in [n^{-1},T]. \tag{3.3}$$

Further, we take a numerical sequence $\{t_i\}$, $i \in \mathbb{N}$ such that

$$t_i \in (0,T),\ t_{i+1} < t_i \text{ for } i \in \mathbb{N} \text{ and } \lim_{i\to\infty} t_i = 0.$$

Clearly, for each $n \in N_1 = \{n \in N_T : n^{-1} < t_1\}$ we have

$$A \le x_n(t) < \tilde{M}_0,\ \ t \in [t_1,T],$$

and

$$m_1 \le x_n'(t) \le M_1,\ \ t \in [t_1,T].$$

Besides, the continuity of f on the set $[t_1,T]\times[A,M_0]\times[m_1,M_1]$ implies that there is a constant $M_{1,2}$, independent on n, such that

$$|x_n''(t)| \le M_{1,2},\ t \in [t_1,T].$$

According to the Arzela - Ascoli theorem, the bounds for $x_n(t)$, $x_n'(t)$ and $x_n''(t)$ on $[t_1,T]$ allow us to conclude that the sequence $\{x_n\}$ of $C^2[n^{-1},T]$-solutions to (3.1) has a subsequence $\{x_{1,n_k}\}$, $k \in \mathbb{N}$ $n_k \in N_1$, converges uniformly on the interval $[t_1,T]$ to some function $x_{t_1}(t) \in C^1[t_1,T]$, i.e. $\| x_{1,n_k} - x_{t_1} \|_1 \to 0$ on $[t_1,T]$. Now (3.2) and (3.3) yield, respectively,

$$A \le x_{t_1}(t) \le \tilde{M}_0,\ \ t \in [t_1,T],$$

and

$$m_1 \le x_{t_1}'(t) \le M_1,\ \ t \in [t_1,T].$$

On the other hand, as a solution of (3.1) each element of $\{x_{1,n_k}\}$ satisfies the integral equation

$$x'_{1,n_k}(t) = x'_{1,n_k}(t_1) + \int_{t_1}^{t} f(s, x_{1,n_k}(s), x'_{1,n_k}(s))ds,\ t \in (t_1,T].$$

Further, using that $f(t,x,p)$ is uniformly continuous on the set $[t_1,T]\times[A,\tilde{M}_0]\times[m_1,M_1]$ and that the sequences $\{x_{1,n_k}\}$ and $\{x'_{1,n_k}\}$ are uniformly convergent on $[t_1,T]$, we establish that the sequence $\{f(t, x_{1,n_k}(t), x'_{1,n_k}(t))\}$, $n_k \in N_1$, converges uniformly on $[t_1,T]$ to

$f(t, x_{t_1}(t), x'_{t_1}(t))$. As a result, we obtain that $x_{t_1}(t)$ is a $C^1[t_1, T]$ - solution to the integral equation

$$x'_{t_1}(t) = x'_{t_1}(t_1) + \int_{t_1}^{t} f(s, x_{t_1}(s), x'_{t_1}(s))ds,\ t \in (t_1, T],$$

from where it follows $x_{t_1}(t)$ is a $C^2[t_1, T]$-solution to the differential equation $x'' = f(t, x, x')$ on the interval $[t_1, T]$.

Similarly, considering consecutively sequences

$$\{x_{i,n_k}\},\ n_k \in N_i = \{n_k \in N_T, k \in \mathbb{N}: n_k^{-1} < t_i\}, i \in \mathbb{N}$$

such that $\{x_{i+1,n_k}\}$ is a subsequence of $\{x_{i,n_k}\}$, $i \in \mathbb{N}$ we establish that for each $i \in \mathbb{N}$ there exists a $x_{t_i}(t)$ which is a $C^2[t_i, T]$-solution to the equation $x'' = f(t, x, x')$ on $[t_i, T]$ and has the properties:

$$\| x_{i,n_k} - x_{t_i} \|_1 \to 0 \text{ on the interval } [t_i, T],$$
$$A \le x_{t_i}(t) \le \tilde{M}_0 \text{ for } t \in [t_i, T],$$
$$m_1 \le x'_{t_i}(t) \le M_1 \text{ for } t \in [t_i, T],$$
$$x_{t_{i+1}}(t) \equiv x_{t_i}(t) \text{ for } t \in [t_i, T].$$

In summary, we conclude that there is a function $x_0(t)$, which is a $C^2(0, T]$-solution to $x'' = f(t, x, x')$ on $(0, T]$,

$$A \le x_0(t) \le \tilde{M}_0 \text{ for } t \in (0, T],$$
$$m_1 \le x'_0(t) \le M_1 \text{ for } t \in (0, T], \quad (3.4)$$
$$x_0(t) \equiv x_{t_i}(t) \text{ for } t \in [t_i, T].$$

Further, exactly as in the proof of Theorem 1.1 we establish that

$$\lim_{t \to 0^+} x_0(t) = A$$

and so the function

$$x(t) = \begin{cases} A, & t = 0 \\ x_0(t), & t \in (0, T], \end{cases}$$

is a $C[0, T] \cap C^2(0, T]$-solution to $x'' = f(t, x, x')$. Besides, using that $x'(t) = x_0{}'(t)$ for $t \in (0, T]$, we get

$$x'(0) = \lim_{t \to 0^+} \frac{x(t) - x(0)}{t - 0} = \lim_{t \to 0^+} x'(t) = \lim_{t \to 0^+} x'_0(t) = B,$$

which means $x' \in C[0, T]$ and so $x(t)$ is a $C^1[0, T] \cap C^2(0, T]$-solution to singular IVP (1.1).

Finally, (3.5) gives

$$m_1 \le x'(t) \le M_1 \text{ for } t \in [0, T], \quad (3.5)$$

and an integration of these inequalities from 0 to $t \in (0, T]$ yields the bounds for $x(t)$.

Example 3.1. Consider the IVP

$$x'' = f(t, x, x'),$$
$$x(0) = 5, \quad x'(0) = 0,$$

where $f(t, x, p) = \begin{cases} t^{-\frac{2m}{n}} \sqrt{x - 4}(10 - p) \text{ for } (t, x, p) \in (\mathbb{R}\backslash\{0\}) \times [4, \infty) \times \mathbb{R} \\ t^{-\frac{2m}{n}} \sqrt{4 - x}\,(10 - p) - p \text{ for } (t, x, p) \in (\mathbb{R}\{0\}) \times (-\infty, 4) \times \mathbb{R} \end{cases}$, and $m, n \in N$.

Obviously, because of the multiplier $t^{-\frac{2m}{n}}$, this problem is singular at $t=0$. It is clear also that here $Dt=\mathbb{R}\backslash\{0\}$, $Dx=Dp=\mathbb{R}$.Choose $m_1=0$ and, for example, $\bar{m}_1=-1$, $M_1=12$, $\bar{M}_1=13$, $\tau=0.1$ and fix an arbitrary $T>0$. Then, $\tilde{M}_0=12T+5$. Now, it is not hard to check that all requirements of $\mathbf{S_1}$ are satisfied. Thus, by Theorem 1.2, this problem has a positive non-decreasing solution in $C^1[0,T]\cap C^2(0,T]$ for each fixed $T>0$.

Remark 3.1. It is worth remarking here that Theorem 1.1 is not applicable to study the above problem since (1.3) of **S** is not satisfied. In deed, studying the solvability of Example 3.1 we have established that $\mathbf{S_1}$ holds for $m_1=0$, $M_1=12$, $\tau=0.1$ and $\tilde{M}_0=12T+5$. On the other hand, **S** will be satisfied for $m_1=0$ and the same constants M_1, τ and $\tilde{M}_0$ if $f(t,x,p)$ is continuous for $(t,x,p)\in(0,T]\times[-12T-5.1,12T+5.1]\times[-0.1,12.1]$. But this is not true since if, for example, $t=T$ and $p=2$ we have

$$\lim_{x\to 4^-} f(T,x,2)=-4 \quad \text{and} \quad \lim_{x\to 4^+} f(T,x,2)=0.$$

The same conclusion is also reached for another admissible choice of M_1.

REFERENCES

[1] Aslanov, A. A singular initial-value problem for second-order differential equations. Abstract and Applied Analysis 2014, doi:10.1155/2014/526549.

[2] Agarwal, R., D. O’Regan. Second order initial value problems of Lane-Emden type. Applied Mathematics Letters 20 (2007), 1198–1205.

[3] Agarwal, R., D. O'Regan. Second-order initial value problems of singular type. J. Math. Anal. Appl. 229 (1999), 441-451.

[4] Bobisud, L., D. O'Regan. Existence of solutions to some singular initial value problems. J. Math. Anal. Appl. 133 (1988), 215-230.

[5] Bobisud, L., Y. Lee. Existence of monotone or positive solutions of singular second-order sublinear differential equations. J. Math. Anal. Appl. 159 (1991), 449-468.

[6] Cabada, A., J. Cid, R. Pouso. Positive solutions for a class of singular differential equations arising in diffusion processes. Dynamics of Continuous, Discrete and Impulsive Systems A 12 (2005), 329-342.

[7] Cabada, A., S. Heikkilä. Extremaly results for discontinuous explicit and implicit diffusion problems. J. Comput. Appl. Math. 143 (2002), 69-80.

[8] Cabada, A., J. Nieto, R. Pouso. Approximate solutions to a new class of nonlinear diffusion problems. J. Comput. Appl. Math. 108 (1999), 219-231.

[9] Cid, J. Extremal positive solutions for a class of singular and discontinuous second order problems. Nonl. Anal. 51 (2002), 1055-1072.

[10] Kelevedjiev, P., N. Popivanov. Second-order initial value problems with singularities. Boundary Value Problems 2014, 2014:161.

[11] Kelevedjiev, P., N. Popivanov, L. Bekesheva. On the solvability of a second-order singular initial value problem. AIP Conference Proceedings 1631, 245(2014); doi: 10.1063/1.4902483.

[12] Kelevedjiev, P. Solvability of singular second-order initial-value problems. Electronic Journal of Differential Equations 2016 (2016), No. 273, 1-8.

[13] Maagli, H., S. Masmoudi. Existence theorem of nonlinear singular boundary value problem. Nonl. Anal. 46 (2001), 465-473.

[14] Rachůnková, I., J. Tomeček. Bubble-type solutions of non-linear singular problem. Mathematical and Computer Modeling 51 (2010), 658-669.

[15] Rachůnková, I., J. Tomeček. Homoclinic solutions of singular nonautonomous second order differential equations. Boundary Value Problems, Vol. 2009, doi: 101155/2009/959636, 21 pages.

[16] Rachůnková, I., J. Tomeček. Strictly increasing solutions of a nonlinear singular differential equation arising in hydrodynamics. Nonlinear Analysis 72 (2010), 2114-2118.

[17] Yang, G. Minimal positive solutions to some singular second-order differential equations. J. Math.Anal. Appl. 266 (2002), 479- 491.

[18] Yang, G. Positive solutions of some second-order nonlinear singular differential equations. Computers and Mathematics with Applications 45 (2003), 605-614.

[19] Zhao, Z. Positive solutions of nonlinear second order ordinary differential equations. Proc. Amer. Math. Soc. 121 (1994), 465-469.

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ВЪРХУ РАЗРЕШИМОСТТА НА СИНГУЛЯРНА НАЧАЛНА ЗАДАЧА ОТ ВТОРИ РЕД

Петьо Келеведжиев, Силвестър Божериков

Технически университет – София, ИПФ – Сливен

Резюме: *Използвайки Топологичната трансверзална теорема и техниката на бариерните ивици, изследваме разрешимостта на начална задача от втори ред, сингулярна в началната стойност на независимата променлива.Полученият резултат гарантира съществуването на поне едно монотонно решение, което не променя знака си.*

Ключови думи: *Начална задача, обикновени диференциални уравнения, сингулярност, съществуване, монотонни и положителни решения, бариерна техника.*

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