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BOOK 5

"MATHEMATICS, INFORMATICS AND PHYSICS"

VOLUME 13

CONTENTS

Mathematics

Informatics

BOOK 5

"MATHEMATICS, INFORMATICS AND PHYSICS"

VOLUME 13

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MULTIPLE SOLUTIONS FOR A NONLINEAR DISCRETE FOURTH ORDER p -LAPLACIAN EQUATION

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Abstract: *We study the existence of multiple solutions for a nonlinear discrete fourth order p-Laplacian equation. The proof of the main results is based on the three critical points theorem due to B. Ricceri. An example is given.*

Keywords: *Fourth order difference equation, p-Laplacian, variational methods.*

INTRODUCTION

Recently, many researchers study fourth order differential and difference equations. Difference equations occur in dynamical systems, statistics, economics, biology and other fields. Many existence results have been established by applying variational methods. Recall for example the papers of Cai and Guo [2], Cai and Yu [3], Deng, Liu, Zhang and Shi [4], Iannizzotto and Tersian [5]. In [6] Graef, Kong and Wang consider the problem

$$\Delta^4 u(t-2) - \Delta\left(p(t-1)\Delta u(t-1)\right) + q(t)u(t) = f(t,u(t)), \quad t = 1,\ldots,N$$

with periodic boundary conditions

$$\Delta^i u(-1) = \Delta^i u(N-1), \quad i = 0,1,2,3.$$

They obtain multiplicity results if the functions p and q are nonnegative and f is odd on its second variable.

In [7] Long and Shi deal with the following discrete boundary value problem

$$-\Delta\phi_p\left(\Delta u(t-1)\right) + \phi_p\left(u(t)\right) = \lambda f(t,u(t)), \quad t = 1,\ldots,T$$

with homogeneous Dirichlet conditions

$$u(0) = u(T+1) = 0,$$

using a critical point theorem, due to Bonanno in [1].

In the present paper we obtain criteria for the existence of three solutions for the fourth order problem

$$\Delta^2\left(\varphi_p\left(\Delta^2 u(t-2)\right)\right) + \alpha\varphi_p\left(u(t)\right) = \lambda f\left(t,u(t)\right), \quad t \in I = \{1,2,\ldots,T\}, T \geq 2 \tag{1}$$

with the boundary condition

$$u(0) = \Delta u(-1) = \Delta^2 u(T) = 0, \quad \Delta\left(\varphi_p\left(\Delta^2 u(T-1)\right)\right) = \mu g(u(T+1)), \tag{2}$$

where α, λ, μ are real parameters, f and g are continuous.

Here $\varphi_p(x) = |x|^{p-2}x,\ p \geq 2,$ Δ is the forward difference operator defined by:

$$\Delta u(t) = u(t+1) - u(t),\ \Delta^k u(t) = \Delta^{k-1}\left(\Delta u(t)\right), k \in N, k \geq 2.$$

We also assume that $\alpha > -\dfrac{1}{T^p(T+1)^p}$.

Our main results in this paper are:

Theorem 1.1. *Assume that the following conditions hold:*

$(H1)$ There exists $A > 0$, such that $\max\{g^0, g^\infty\} < A,$ where

$$g^0 = \lim_{y\to 0}\sup \frac{-\int_0^y g(s)ds}{y^p} \quad \text{and} \quad g^\infty = \lim_{|y|\to\infty}\sup \frac{-\int_0^y g(s)ds}{y^p};$$

$(H2)$ There exists $k>0,$ such that

$$pAB\rho^p < -\int_0^{k(T+1)^2} g(s)ds,$$

where

$$B = \frac{k^p}{p}\left(2^p(T+1)+\alpha\sum_{t=1}^{T} t^{2p}\right) > 0.$$

Then, for each interval $[a,b],$ satisfying

$$[a,b]\subset\left(\frac{B}{-\int_0^{k(T+1)^2} g(s)ds}, \frac{1}{pA\rho^p}\right)$$

there exists $K>0,$ such that for every $\mu\in$ $[a,b],$ there exists $\xi>0$, such that for each $\lambda\in[0,\xi]$ the boundary problem (1),(2) has at least three solutions in $X,$ whose norms are less than K;

Theorem 1.2. *Assume that the following conditions hold:*
(H3) There exists $0<C<\infty,$ such that

$$\max\left\{\lim_{y\to 0}\sup\frac{\max_{t\in I} F(t,y)}{y^p}, \lim_{|y|\to\infty}\sup\frac{\max_{t\in I} F(t,y)}{y^p}\right\} < C,$$

(H4) There exists $\sigma>0,$ such that

$$pCDT\rho^p < \sum_{t=1}^{T}\int_0^{\sigma t^2} f(t,s)ds,$$

where

$$D = \frac{\sigma^p}{p}\left(2^p(T+1)+\alpha\sum_{t=1}^{T} t^{2p}\right) > 0.$$

Then, for each interval

$$[a,b]\subset\left(\frac{D}{\sum_{t=1}^{T}\int_0^{\sigma t^2} f(t,s)ds}, \frac{1}{pCT\rho^p}\right),$$

there exists $K>0$, such that for every $\lambda\in[a,b],$ there exists $\xi>0$ such that for each $\mu\in[0,\xi],$ the boundary problem (1),(2) has at least three solutions in X, whose norms are less than $K.$

This paper is organized as follows. In the next Section we present the variation formulation of the problem and some lemmas. In the last Section the proofs of the main results and an example are given.

VARIATIONAL FRAMEWORK AND PRELIMINARY LEMMAS

In this section we give some auxiliary results for the variational formulation and treatment of the problem. Let us define the real vector space

$$X := \{u : \{-1,\ldots,T+2\} \to R : u(0) = \Delta u(-1) = \Delta^2 u(T) = 0\}$$

with norm

$$\|u\|_X = \left(\sum_{t=1}^{T+1} \left|\Delta^2 u(t-2)\right|^p + \alpha \sum_{t=1}^{T} |u(t)|^p \right)^{\frac{1}{p}}$$

and denote

$$\rho = (T+1)^{\frac{2p-1}{p}} \left(1 + \min\{\alpha, 0\} T^p (T+1)^p\right)^{-\frac{1}{p}}.$$

Lemma 2.1. *For every* $u \in X$ *we have*

$$\sum_{t=1}^{T+1} \left|\Delta^2 u(t-2)\right|^p + \alpha \sum_{t=1}^{T} |u(t)|^p \geq 0$$

and

$$|u(t)| \leq \rho \|u\|_X \quad \text{for every} \quad t \in I.$$

Proof: Let $u \in X$ and $t \in I$. One can check that

$$\Delta u(t-1) = \Delta u(-1) + \sum_{i=1}^{t} \Delta^2 u(i-2) = \sum_{i=1}^{t} \Delta^2 u(i-2).$$

Using the above result and Hölder's inequality, we obtain that

$$\left|\Delta u(t-1)\right| \leq \sum_{i=1}^{t} \left|\Delta^2 u(i-2)\right| \leq \sum_{i=1}^{T+1} \left|\Delta^2 u(i-2)\right|$$

$$\leq (T+1)^{\frac{p-1}{p}} \left(\sum_{i=1}^{T+1} \left|\Delta^2 u(i-2)\right|^p \right)^{\frac{1}{p}}.$$

and

$$\left|\Delta u(t-1)\right|^p \leq (T+1)^{p-1} \sum_{i=1}^{T+1} \left|\Delta^2 u(i-2)\right|^p.$$

Consequently,

$$\sum_{t=1}^{T+1} \left|\Delta u(t-1)\right|^p \leq (T+1)^p \sum_{t=1}^{T+1} \left|\Delta^2 u(t-2)\right|^p. \tag{3}$$

Note that

$$u(t) = u(0) + \sum_{i=1}^{t} \Delta u(i-1) = \sum_{i=1}^{t} \Delta u(i-1).$$

Then from (3) and Hölder's inequality, for every $t \in \{1,\ldots,T+1\}$ we have

$$\begin{aligned} |u(t)| &\leq \sum_{i=1}^{t} \left|\Delta u(i-1)\right| \leq (T+1)^{\frac{p-1}{p}} \left(\sum_{i=1}^{T+1} \left|\Delta^2 u(i-1)\right|^p \right)^{\frac{1}{p}} \\ &\leq (T+1)^{\frac{2p-1}{p}} \left(\sum_{i=1}^{T+1} \left|\Delta^2 u(i-2)\right|^p \right)^{\frac{1}{p}}. \end{aligned} \tag{4}$$

If $t \in I$

$$|u(t)| \le \sum_{i=1}^{t} |\Delta u(i-1)| \le T^{\frac{p-1}{p}} \left(\sum_{i=1}^{T} |\Delta u(i-1)|^p \right)^{\frac{1}{p}},$$

$$|u(t)|^p \le T^{p-1} \sum_{i=1}^{T} |\Delta u(i-1)|^p$$

and

$$\sum_{t=1}^{T} |u(t)|^p \le T^p \sum_{i=1}^{T} |\Delta u(i-1)|^p \le T^p \sum_{i=1}^{T+1} |\Delta u(i-1)|^p.$$

From (3) we obtain that

$$\sum_{t=1}^{T} |u(t)|^p \le T^p (T+1)^p \sum_{t=1}^{T+1} |\Delta^2 u(t-2)|^p. \tag{5}$$

Using (3) and (5) we have

$$\begin{aligned} &\sum_{t=1}^{T+1} |\Delta^2 u(t-2)|^p + \alpha \sum_{t=1}^{T} |u(t)|^p \\ &\ge \left(1 + \min\{\alpha, 0\} T^p (T+1)^p\right) \sum_{t=1}^{T+1} |\Delta^2 u(t-2)|^p \\ &\ge 0. \end{aligned} \tag{6}$$

Thus we proved the first statement of Lemma 2.1. From (6) we have

$$\|u\|_X \ge \left(1 + \min\{\alpha, 0\} T^p (T+1)^p\right)^{\frac{1}{p}} \left(\sum_{t=1}^{T+1} |\Delta^2 u(t-2)|^p \right)^{\frac{1}{p}}$$

and

$$\rho \|u\|_X \ge |u(t)|$$

for every $t \in I$, which completes the proof of the Lemma 2.1.

Remark 2.1. *It is clear that X is $T+1$-dimentional Euclidean space with norm $\|.\|_X$.*

For every $u \in X$ let us define the functionals

$$\Phi(u) = \frac{1}{p} \|u\|_X^p, \tag{7}$$

$$J(u) = -\int_0^{u(T+1)} g(s)\,ds \tag{8}$$

and

$$\Psi(u) = \sum_{t=1}^{T} \int_0^{u(t)} f(t,s)\,ds. \tag{9}$$

They are well defined and continuously differentiable, whose derivatives are

$$\Phi'(u)(v) = \sum_{t=1}^{T+1} \Delta^2 u(t-2) |\Delta^2 u(t-2)|^{p-2} \Delta^2 v(t-2) + \alpha \sum_{t=1}^{T} u(t) |u(t)|^{p-2} v(t),$$

$$J'(u)(v) = -g(u(T+1)) v(T+1)$$

and

$$\Psi'(u)(v) = \sum_{t=1}^{T} f(t, u(t)) v(t)$$

for every $v \in X$.

Lemma 2.2. *For every* $u,v\in X$, *we have*

$$\sum_{t=1}^{T+1}\Delta^2u(t-2)\left|\Delta^2u(t-2)\right|^{p-2}\Delta^2v(t-2)$$
$$=-\Delta\left(\Delta^2u(T-1)\left|\Delta^2u(T-1)\right|^{p-2}\right)v(T+1)$$
$$+\sum_{t=1}^{T}\Delta^2\left(\Delta^2u(t-2)\left|\Delta^2u(t-2)\right|^{p-2}\right)v(t).$$

Proof: Using the boundary conditions $v(0)=\Delta v(-1)=\Delta^2u(T)=0$ and by the summation by parts formula it follows that

$$\sum_{t=1}^{T+1}\Delta^2u(t-2)\left|\Delta^2u(t-2)\right|^{p-2}\Delta^2v(t-2)$$
$$=\Delta^2u(T)\left|\Delta^2u(T)\right|^{p-2}\Delta v(T)-\Delta^2u(-1)\left|\Delta^2u(-1)\right|^{p-2}\Delta v(-1)$$
$$-\sum_{t=1}^{T+1}\Delta\left(\Delta^2u(t-2)\left|\Delta^2u(t-2)\right|^{p-2}\right)\Delta v(t-1)$$
$$=-\sum_{t=1}^{T+1}\Delta\left(\Delta^2u(t-2)\left|\Delta^2u(t-2)\right|^{p-2}\right)\Delta v(t-1)$$
$$=-\Delta\left(\Delta^2u(T-1)\left|\Delta^2u(T-1)\right|^{p-2}\right)\Delta v(T)$$
$$-\sum_{t=1}^{T}\Delta\left(\Delta^2u(t-2)\left|\Delta^2u(t-2)\right|^{p-2}\right)\Delta v(t-1)$$
$$=-\Delta\left(\Delta^2u(T-1)\left|\Delta^2u(T-1)\right|^{p-2}\right)\Delta v(T)$$
$$-\Delta\left(\Delta^2u(T-1)\left|\Delta^2u(T-1)\right|^{p-2}\right)v(T)+\Delta\left(\Delta^2u(-1)\left|\Delta^2u(-1)\right|^{p-2}\right)v(0)$$
$$+\sum_{t=1}^{T}\Delta^2\left(\Delta^2u(t-2)\left|\Delta^2u(t-2)\right|^{p-2}\right)v(t)$$
$$=-\Delta\left(\Delta^2u(T-1)\left|\Delta^2u(T-1)\right|^{p-2}\right)v(T+1)$$
$$+\sum_{t=1}^{T}\Delta^2\left(\Delta^2u(t-2)\left|\Delta^2u(t-2)\right|^{p-2}\right)v(t)$$

Lemma 2.3. *If* $u\in X$ *is a critical point of the functional* $\Phi-\lambda\Psi-\mu J$, *then* u *is a solution of the boundary value problem (3),(5).*

Proof: Let $u\in X$ be a critical point of the functional $\Phi-\lambda\Psi-\mu J$. Then

$$\Phi'(u)(v)-\lambda\Psi'(u)(v)-\mu J'(u)(v)=0 \quad \text{for every } v\in X.$$

We have that

$$\Phi'(u)(v)-\lambda\Psi'(u)(v)-\mu J'(u)(v)$$
$$=\sum_{t=1}^{T+1}\Delta^2u(t-2)\left|\Delta^2u(t-2)\right|^{p-2}\Delta^2v(t-2)+\alpha\sum_{t=1}^{T}u(t)\left|u(t)\right|^{p-2}v(t)$$
$$-\lambda\sum_{t=1}^{T}f(t,u(t))v(t)+\mu g(u(T+1))v(T+1)$$

$$= v(T+1)\left(\mu g(u(T+1)) - \Delta\left(\Delta^2 u(T-1)\left|\Delta^2 u(T-1)\right|^{p-2}\right)\right)$$
$$+ v(t)\left(\sum_{t=1}^{T}\Delta^2\left(\varphi_p\left(\Delta^2 u(t-2)\right)\right) + \alpha\sum_{t=1}^{T}\varphi_p(u(t)) - \lambda\sum_{t=1}^{T} f(t,u(t))\right)$$
$$= 0.$$

Thus, by the arbitrariness of $v \in X$, we have that

$$\Delta\left(\varphi_p\left(\Delta^2 u(T-1)\right)\right) = \mu g(u(T+1))$$

and

$$\Delta^2\left(\varphi_p\left(\Delta^2 u(t-2)\right)\right) + \alpha\varphi_p(u(t)) = \lambda f(t,u(t)) \quad \text{for every } t \in I.$$

The critical poitn u also satisfies the boundary conditions, so u is a solution of the boundary value problem (3),(5).

Let E be a real Banach with a norm $\|.\|_E$ and let W_E be the class of all functionals $\Phi : E \to R$ with the property: if $\{u_n\}$ is a sequence in E converging weakly to u and $\lim_{n\to\infty}\inf \Phi(u_n) \le \Phi(u),$ then $\{u_n\}$ has a subsequence converging strongly to u .

Recall a three critical points theorem due to Ricceri [9, Theorem 2].

Theorem 2.1. *Let E be a separable and reflexive real Banach space with the norm $\|.\|_E$, and $\Phi : E \to R$ be a coercive, sequentially weakly lower semicontinuous C^1 functional, belonging to W_E , bounded on each bounded subset of E and whose derivative admits a continuous inverse on E^*. Assume that $J : E \to R$ is a C^1 functional with compact derivative. Let Φ has a strict local minimum $u_0,\ \Phi(u_0) = J(u_0) = 0,$*

$$\delta = \max\left\{0, \lim_{\|x\|_E\to\infty}\sup\frac{J(u)}{\Phi(u)}, \lim_{\|x\|_E\to 0}\sup\frac{J(u)}{\Phi(u)}\right\},$$
$$\eta = \sup_{u\in\Phi^{-1}(0,\infty)}\frac{J(u)}{\Phi(u)}$$

and $\delta < \eta.$

Then, for each interval $[a,b] \subset \left(\frac{1}{\eta}, \frac{1}{\delta}\right)$, there exists $K > 0$ with the following property:

for every $x \in [a,b]$ and every C^1 functional $\Psi : E \to R$ with compact derivative, there exists $\xi > 0$, such that for each $y \in [0,\xi]$ the equation

$$\Phi'(u) - y\Psi'(u) - xJ'(u) = 0$$

has at least three solutions in E , whose norms are less than $K.$

PROOF OF THE MAIN RESULTS

This section is devoted to proof the main results of this paper. Moreover, we present an example to illustrate these results.

Proof of Theorem 1.1: Let the functionals $\Phi, J, \Psi : X \to R$ are defined by (7)-(9). They are continuously differentiable and their derivatives are calculated above.

From Lemma 2.3, every critical point of the functional $\Phi - \lambda\Psi - \mu J$ is a solution of (3),(5). We apply Theorem 2.1 with $E = X, x = \mu$ and $y = \lambda.$

First, we show that the assumptions in Theorem 2.1 are satisfied.

Φ is a coercive and sequentially weakly lower semicontinuous C^1 functional and is bounded on each bounded subset of X.

For every $0 \neq u \in X$ we have $\Phi'(u)(u) = \|u\|_X^p$ and $\lim_{\|u\|_X \to \infty} \frac{\Phi'(u)(u)}{\|u\|_X} = \infty$. Thus Φ' is coercive. Moreover, by Lemma A.0.5 in [8], there exists $c_p > 0$, such that

$$\left(\Phi'(u) - \Phi'(v)\right)(u - v) \geq c_p \|u - v\|_X^p \quad \text{for all} \quad u, v \in X.$$

Therefore, Φ' is uniformly monotone. Now by [10] $\left(\Phi'\right)^{-1} : X^* \to X$ exists and is continuous. One can check that J' and Ψ' are compact operators. Let $u_0 \equiv 0$. Then Φ has a strict local minimum and $\Phi(u_0) = J(u_0) = 0$.

Now we will show that

$$\delta < pA\rho^p \text{and} \frac{-\int_0^{k(T+1)^2} g(s)ds}{B} \leq \eta.$$

From (H1) there exist $0 < \gamma_1 < \gamma_2$ such that

$$-\int_0^y g(s)ds \leq Ay^p \quad \text{for all} \quad |y| \in 0, \gamma_1) \cup (\gamma_2, \infty).$$

This, together with the continuity of g implies that there exists $c_1 > 0$ and $\theta > p$, such that

$$-\int_0^y g(s)ds \leq A|y|^p + c_1|y|^\theta \quad \text{for all} \quad y \in R.$$

Using Lemma 2.1 it follows that

$$J(u) \leq A|u(T+1)|^p + c_1|u(T+1)|^\theta$$

$$\leq A\rho^p \|u\|_X^p + c_1\rho^\theta \|u\|_X^\theta \quad \text{for all} \quad u \in X.$$

Hence

$$\lim_{u \to 0} \sup \frac{J(u)}{\Phi(u)} \leq pA\rho^p.$$

If $u(T+1) \leq \gamma_2$, then for every $u \in X$ we have $J(u) = -\int_0^{u(T+1)} g(s)ds \leq c_2$ for some $c_2 > 0$. If $u(T+1) > \gamma_2$, then $J(u) = -\int_0^{u(T+1)} g(s)ds \leq A|u(T+1)|^p$. Thus

$$J(u) \leq c_2 + A|u(T+1)|^p \leq c_2 + A\rho^p \|u\|_X^p.$$

Then

$$\lim_{u \to \infty} \sup \frac{J(u)}{\Phi(u)} \leq pA\rho^p$$

and $\delta \leq pA\rho^p$.

Let k is given in (H2) and we define $u_1(t) = kt^2$. One can check that

$$\Delta u_1(t-1) = k(2t-1) \quad \text{and} \quad \Delta^2 u_1(t-2) = 2k.$$

Using Lemma 2.1 we obtain that

$$\Phi(u_1) = \frac{k^p}{p}\left(2^p(T+1) + \alpha \sum_{t=1}^{T} t^{2p}\right) = B > 0.$$

Then $u_1 \in \Phi^{-1}(0, \infty)$ and we have

$$\eta = \sup_{u\in\Phi^{-1}(0,\infty)} \frac{J(u)}{\Phi(u)} \geq \frac{J(u_1)}{\Phi(u_1)} = \frac{-\int_0^{k(T+1)^2} g(s)ds}{B}.$$

Hence, all the assumptions of Lemma 2.1 are satisfied and the assertion follows.

Proof of Theorem 1.2: Let the functionals $\Phi, J, \Psi : X \to R$ are defined by (7)-(9). By Lemma 2.3 every critical point of the functional $\Phi - \lambda\Psi - \mu J$ is a solution of (1),(2). Thus, to prove Theorem 1.2, we will apply Theorem 2.1 with $E = X$, $J = \Phi$, $\Psi = J$, $x = \lambda$ and $y = \mu$.

As in the proof of Theorem 1.1, Φ, Ψ and J satisfy the assumptions of Theorem 2.1. Moreover, for $u_0 = 0$, Φ has a strict local minimum and $\Phi(u_0) = \Psi(u_0) = 0$.

We will show that

$$\delta \leq pCT\rho^p \quad \text{and} \quad \sum_{t=1}^{T}\int_0^{\sigma t^2} f(t,s)ds \leq \eta D.$$

From (H3), there exist $0 < \tau_1 < \tau_2$, such that

$$F(t,y) \leq C|y|^p \text{ for all } t \in I \text{ and } \quad |y| \in [0,\tau_1) \cup (\tau_2,\infty).$$

This, together with the continuity of f implies that there exists $c_3 > 0$ and $\vartheta > p$, such that

$$F(t,y) \leq C|y|^p + c_3|y|^\vartheta \quad \text{for all} \quad t \in I \quad \text{and} \quad |y| \in [0,\tau_1) \cup (\tau_2,\infty).$$

Then

$$\Psi(u) \leq \sum_{t=1}^{T}\left(C|u(t)|^p + c_3|u(t)|^\vartheta\right)$$

$$\leq CT\rho^p \|u\|_X^p + c_3T\rho^\vartheta \|u\|_X^\vartheta \quad \text{for all} \quad u \in X.$$

Hence,

$$\lim_{u\to 0}\sup \frac{\Psi(u)}{\Phi(u)} \leq pCT\rho^p.$$

If for all $u \in X$ and $t \in I$, $u(t) \leq \tau_2$, then $F(t,u(t)) \leq c_4$ for some $c_4 > 0$. If $u(t) \geq \tau_2$, then $F(t,u(t)) \leq C|u(t)|^p$. Thus

$$\Psi(u) \leq c_4T + CT|u(t)|^p \leq c_4T + CT\rho^p\|u\|_X^p.$$

Then

$$\lim_{u\to\infty}\sup \frac{\Psi(u)}{\Phi(u)} \leq pCT\rho^p$$

and $\delta \leq pCT\rho^p$.

Let σ is given in (H4) and define $u_2(t) = \sigma t^2$. Then

$$\Phi(u_2) = \frac{\sigma^p}{p}\left(2^p(T+1) + \alpha\sum_{t=1}^{T} t^{2p}\right) = D > 0.$$

It follows that $u_2 \in \Phi^{-1}(0,\infty)$,

$$\eta = \sup_{u\in\Phi^{-1}(0,\infty)} \frac{\Psi(u)}{\Phi(u)} \geq \frac{\Psi(u_2)}{\Phi(u_2)} = \frac{\sum_{t=1}^{T}\int_0^{\sigma t^2} f(t,s)ds}{D}$$

and $\delta < \eta$. Hence, all the assumptions of Theorem 2.1 are satisfied.

We present an example to illustrate Theorem 1.1.

Example 3.1. *Let*

$$g(y)=\begin{cases}3, & y<-1,\\ 3y^2, & -1\le y<0,\\ -3y^2, & 0\le y\le 1,\\ -3, & y>1.\end{cases}$$

We claim that for each interval $[a,b]\subset\left(\frac{116}{27^2},\infty\right)$, there exists $K>0$, such that for every $\mu\in[a,b]$, there exists $\xi>0$, such that for each $\lambda\in[0,\xi]$ the problem

$$\Delta^4u(t-2)+u(t)=\lambda f(t,u(t)),\ t=1,2$$

with the boundary conditions

$$u(0)=\Delta u(-1)=\Delta^2u(2)=0,\Delta^3u(1)=\mu g(u(3)),$$

where f is continuous, has at least three solutions in X, whose norms are less than K.

Proof: Let us choose $\alpha=1, p=T=2.$ By the definition of g it follows that

$$\int_0^y g(s)ds=\begin{cases}-|y|^3, & |y|\le 1,\\ -3|y|+2, & |y|>1.\end{cases}$$

Then

$$g^0=\limsup_{y\to 0}\frac{-\int_0^y g(s)ds}{y^2}=0 \quad\text{and}\quad g^\infty=\limsup_{|y|\to\infty}\frac{-\int_0^y g(s)ds}{y^2}=0.$$

Thus (H1) holds for any $A>0,$ and (H2) is satisfied for any $k>0.$
Let us choose $k>0,$ such that

$$\frac{B}{-\int_0^{(T+1)^2k}g(s)ds}=\frac{B}{-\int_0^{9k}g(s)ds}$$

has the smallest value, where B is defined in 1.1. For every $k>0$ we have

$$B=\frac{29k^2}{2}.$$

By the definition of $g,$ if $k\in\left(0,\frac{1}{9}\right]$ we have

$$-\int_0^{9k}g(s)ds=\int_0^{9k}3s^2ds=(9k)^3$$

and if $k>\frac{1}{9}$

$$-\int_0^{9k}g(s)ds=\int_0^1 3s^2ds+\int_1^{9k}3ds=27k-2.$$

Then

$$\frac{B}{-\int_0^{9k} g(s)ds} = \begin{cases} \frac{29}{2.9^3 k}, & \text{if } k \in \left(0, \frac{1}{9}\right], \\ \frac{29k^2}{2(27k-2)}, & \text{if } k > \frac{1}{9}. \end{cases}$$

One can check that the smallest value of the above function of k is $\frac{116}{27^2}$, attained at $k = \frac{4}{27}$. Choosing $k = \frac{4}{27}$ for all $A > 0$ the conditions (H1) and (H2) from Theorem 1.1 are satisfied.

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REFERENCES

[1] G. Bonanno, A critical points theorem and nonlinear differential problems, Journal of Global Optimization, vol. 28, 249-258, 2004.

[2] X. Cai and Z. Guo, Existence of solutions of nonlinear fourth order discrete boundary value problem, J. Difference Equ. Appl. 12, 459-466, 2006.

[3] X. Cai and J. Yu, Existence of periodic solutions for a 2nth-order nonlinear difference equation, J. Math. Anal. and Appl., vol 329, 870-878, 2007.

[4] X. Deng, X. Liu, Y. Zhang and H. Shi, Periodic and subharmonic solutions for a 2nth-order difference equation involving p-Laplacian, Indagationes Mathematicae, vol. 24, 613-625, (2013).

[5] A. Iannizzotto and S. A. Tersian, Multiple homoclinic solutions for discrete p-Laplacian via critical point theory, J. Math. Anal. and Appl., vol 403, 173-182, 2013.

[6] J. R. Graef, L. Kong, M. Wang, Multiple solutions to a periodic boundary value problem for a nonlinear discrete fourth order equation, Adv. Dyn. Syst. Appl. 8(3), 203-215, 2013.

[7] Y. Long and H. Shi, Multiple Solutions for the Discrete p-Laplacian Boundary Value Problems, Discrete Dynamics in Nature and Society, 2014, Article ID 213702, 2014.

[8] I. Peral, Multiplicity of solutions for the p-Laplacian, ICTP SMR 990/1, 1997.

[9] B. Ricceri, A further three critical points theorem, Nonlinear Anal. 71, 2009.

[10] E. Zeidler, Nonlinear Functional Analysis and its Applications, Vol. II A and B, Springer-Verlag, New York, 1990.

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