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HIGH-ORDER DIFFERENCE SCHEMES BASED ON INTEGRAL IDENTITIES FOR SYSTEMS OF WEAKLY COUPLED ORDINARY DIFFERENTIAL EQUATIONS

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***Abstract**: High-order finite difference approximations of the solutions of systems of reaction-diffusion equations are constructed. Explicit formulas based on the integral identities that give $O(h^2), O(h^4),\ldots$ accuracy are derived. Three-point schemes of any prescribed order of accuracy are developed using special numerical integration. A rigorous rate of convergence analysis is presented. Numerical experiments confirm the theoretical results.*

***Keywords:** Systems of reaction-diffusion equations, integral identities, finite difference schemes*

INTRODUCTION

In this paper, we examine a high-accurate numerical method for system of ordinary differential equations (ODEs). Such systems of equations have many applications in physics, mechanics, engineering, etc.

The two-point boundary value problems are the simplest, but sufficiently representative models on the example on which can be described many aspects of a numerical method. We consider the system of reaction-diffusion equations

$$\mathrm{L}U := -(P(x)U')' + Q(x)U = F(x) \quad on \quad (0,1) \tag{1}$$

with Dirichlet boundary conditions

$$U(0) = U_0,\ U(1) = U_1, \tag{2}$$

where $\mathrm{L} = (L_1,\ldots,L_m)^T$, $U(x) = (u_1(x),\ldots,u_m(x))^T$, $U'(x) = (u_1(x),\ldots,u_m(x))^T$,

$P(x) = diag(p_1(x),\ldots,p_m(x))$, $Q(x) = (q_{ij}(x))_{m\times m}$, $F(x) = (f_1(x),\ldots,f_m(x))^T$, $m \in N$.

Suppose that matrix $Q(x) = (q_{ij}(x))_{m\times m}$ is an M - matrix (i.e. off - diagonals are non-positive and diagonals is positive) with

$$\min_{x\in[0,1],i=\overline{1,m}} \left\{\sum_{j=1}^{m} q_{ij}(x)\right\} \ge \alpha^2 \ge 0.$$

We suppose that $p_i(x), q_{ij}(x), i,\ j = \overline{1,m}$ are piecewise continuous on $[0,1]$, which can have single discontinuity at a $x = \alpha$, $0 < \alpha < 1$. Then, if we assume U continuous at $x = \alpha$, a formal integration of each equation of (1) gives for $i = \overline{1,m}$:

$$L_i U = -(p_i(x)u_i'(x)) + \sum_{j=1}^{m} q_{ij}(x)u_i(x) = f_i(x),\ i = \overline{1,m}\ on\ (0,\alpha)\cup(\alpha,1), \tag{3}$$

$$[u_i]_\alpha = u_i(\alpha+0) - u_i(\alpha-0) = 0,\ [p_i u_i']_\alpha = 0. \tag{4}$$

Certain of the results that follow will require greater smoothness of the data to the left and right of $x = \alpha$. It is also assumed that

$$0 < p_{0i} \le p_i(x) \le p_{1i},\ x \in [0,1],\ i = \overline{1,m}. \tag{5}$$

Let Q_α^k be the set of continuous functions $v \in Q_\alpha^k$, that are defined on $[0,1]$ and have piecewise derivatives up to order k, where k is an integer. The function v and its derivatives can have bounded discontinuities only at the point α.

The rest of the paper is organized as follows. In Section 2 we derive exact relations of Marchuk type [1] satisfied by the solution of the system (1). On this base, in Section 3 we derive the difference schemes. In Section 4 we study for convergence high-order difference schemes approximated the problem (1)-(5). Numerical experiments are given in Section 5.

INTEGRAL IDENTITIES

We rewrite the problem (1) as follows:

$$-W'(x)+Q(x)U(x)=F(x), x\in(0,1), \tag{6}$$

where $W(x)=P(x)U'(x)$ is the vector flux.

Let introduce on $[0,1]$ the set of mesh points: $x_0=0$, $x_i=x_{i-1}+h_i$, $h_i>0$, $i=1,\ldots,N,\ \ x_N=1,\ \hbar_1=\frac{1}{2}h_1, \hbar_i=\frac{1}{2}(h_i+h_{i+1}), i=2,\ldots,N-1, \hbar_N=\frac{1}{2}h_N$.

From (6) we have

$$U_{j+1}=U_j+\int_{x_j}^{x_{j+1}}U'(x)dx=U_j+\int_{x_j}^{x_{j+1}}P^{-1}(x)W(x)dx=U_j-\int_{x_j}^{x_{j+1}}W(x)*d\Phi_j^{(1)}(x),$$

$$U_{j-1}=U_j-\int_{x_{j-1}}^{x_j}U'(x)dx=U_j-\int_{x_{j-1}}^{x_j}P^{-1}(x)W(x)dx=U_j-\int_{x_{j-1}}^{x_j}W(x)*d\Phi_j^{(1)}(x),$$

where $U_j=U(x_j)$, $\int_a^b Q(x)dx=\left(\int_a^b q_{ij}(x)dx\right)_{m\times m}$, $U(x)V(x)$ is multiplication of a row with a column while $V(x)*U(x)$ is multiplication of a column with a row. $U(x)V(x)=V(x)*U(x)$ and

$$\Phi_j^{(1)}(x)=diag\left(\phi_{1j}^{(1)}(x),\ldots,\phi_{mj}^{(1)}(x)\right),\ \phi_{ij}^{(1)}(x)=\begin{cases}\int_{x_{j-1}}^{x}\frac{dt}{p_i(t)}, x\in(x_{j-1},x_j),\\ \int_{x}^{x_{j+1}}\frac{dt}{p_i(t)}, x\in(x_j,x_{j+1}),\\ 0, x\notin[x_{j-1},x_{j+1}],\\ i=\overline{1,m},\ j=\overline{1,N-1}.\end{cases}$$

An integration by parts with using of (6), implies:

$$U_{j+1}=U_j+\Phi_j^{(1)}(x_j+0)W_j+\int_{x_j}^{x_{j+1}}\Phi_j^{(1)}(x)(QU-F)(x)dx, \tag{7}$$

$$U_{j-1}=U_j-\Phi_j^{(1)}(x_j-0)W_j+\int_{x_{j-1}}^{x_j}\Phi_j^{(1)}(x)(QU-F)(x)dx. \tag{8}$$

Now, from (7), (8) and the relation $[W]_{x_j}=\vec{0}$, we eliminate W_j to obtain

$$\frac{U_j-U_{j+1}}{\Phi_j^{(1)}(x_j+0)}+\frac{U_j-U_{j-1}}{\Phi_j^{(1)}(x_j-0)}$$
$$+\int_{x_{j-1}}^{x_j}\frac{\Phi_j^{(1)}(x)}{\Phi_j^{(1)}(x_j-0)}Q(x)U(x)dx+\int_{x_j}^{x_{j+1}}\frac{\Phi_j^{(1)}(x)}{\Phi_j^{(1)}(x_j+0)}Q(x)U(x)dx \tag{9}$$
$$=\int_{x_{j-1}}^{x_j}\frac{\Phi_j^{(1)}(x)}{\Phi_j^{(1)}(x_j-0)}F(x)dx+\int_{x_j}^{x_{j+1}}\frac{\Phi_j^{(1)}(x)}{\Phi_j^{(1)}(x_j+0)}F(x)dx,$$

which we shall further refer as first integral identity, $\left(\frac{U}{V}=V^{-1}U\right)$.

We let

$$\Phi_j^{(2)}(x)=\begin{cases}\int\limits_{x_{j-1}}^{x}\Phi_j^{(1)}(t)Q(t)\,dt,\ x\in(x_{j-1},x_j),\\ \int\limits_{x}^{x_{j+1}}\Phi_j^{(1)}(t)Q(t)\,dt,\ x\in(x_j,x_{j+1}),\\ 0,\ x\notin[x_{j-1},x_{j+1}],\\ j=\overline{1,N-1}.\end{cases}$$

Then

$$\int\limits_{x_j}^{x_{j+1}}\Phi_j^{(1)}(x)Q(x)U(x)dx=-\int\limits_{x_j}^{x_{j+1}}U(x)*d\Phi_j^{(2)}(x)=\Phi_j^{(2)}(x_j+0)U_j+\int\limits_{x_j}^{x_{j+1}}\Phi_j^{(2)}(x)P^{-1}(x)W(x)dx, \quad (10)$$

$$\int\limits_{x_{j-1}}^{x_j}\Phi_j^{(1)}(x)Q(x)U(x)dx=\int\limits_{x_{j-1}}^{x_j}U(x)*d\Phi_j^{(2)}(x)=\Phi_j^{(2)}(x_j-0)U_j+\int\limits_{x_{j-1}}^{x_j}\Phi_j^{(2)}(x)P^{-1}(x)W(x)dx. \quad (11)$$

We insert (10) in (7) and (11) in (8) to get

$$\begin{aligned}U_{j+1}&=\left(E_m+\int\limits_{x_j}^{x_{j+1}}\Phi_j^{(1)}(x)Q(x)dx\right)U_j+\Phi_j^{(1)}(x_j+0)W_j\\&-\int\limits_{x_j}^{x_{j+1}}\Phi_j^{(1)}(x)F(x)dx+\int\limits_{x_j}^{x_{j+1}}\Phi_j^{(2)}(x)P^{-1}(x)W(x)dx,\end{aligned} \quad (12)$$

$$\begin{aligned}U_{j-1}&=\left(E_m+\int\limits_{x_{j-1}}^{x_j}\Phi_j^{(1)}(x)Q(x)dx\right)U_j-\Phi_j^{(1)}(x_j-0)W_j\\&-\int\limits_{x_{j-1}}^{x_j}\Phi_j^{(1)}(x)F(x)dx-\int\limits_{x_{j-1}}^{x_j}\Phi_j^{(2)}(x)P^{-1}(x)W(x)dx,\end{aligned} \quad (13)$$

where E_m is identity matrix. The elimination of W_j from the last two equations leads to the second integral identity:

$$\begin{aligned}&\frac{U_j-U_{j+1}}{\Phi_j^{(1)}(x_j+0)}+\frac{U_j-U_{j-1}}{\Phi_j^{(1)}(x_j-0)}+\left(\int\limits_{x_{j-1}}^{x_j}\frac{\Phi_j^{(1)}(x)}{\Phi_j^{(1)}(x_j-0)}Q(x)dx+\int\limits_{x_j}^{x_{j+1}}\frac{\Phi_j^{(1)}(x)}{\Phi_j^{(1)}(x_j+0)}Q(x)dx\right)U_j\\&-\int\limits_{x_{j-1}}^{x_j}\frac{\Phi_j^{(2)}(x)}{\Phi_j^{(1)}(x_j-0)}P^{-1}(x)W(x)dx+\int\limits_{x_j}^{x_{j+1}}\frac{\Phi_j^{(2)}(x)}{\Phi_j^{(1)}(x_j+0)}P^{-1}(x)W(x)dx\\&=\int\limits_{x_{j-1}}^{x_j}\frac{\Phi_j^{(1)}(x)}{\Phi_j^{(1)}(x_j-0)}F(x)dx+\int\limits_{x_j}^{x_{j+1}}\frac{\Phi_j^{(1)}(x)}{\Phi_j^{(1)}(x_j+0)}F(x)dx.\end{aligned} \quad (14)$$

This process can be continued. For this we introduce for $n=1,2,\ldots$:

$$\Phi_j^{(2n+1)}(x)=\begin{cases}\int\limits_{x_{j-1}}^{x}\Phi_j^{(2n)}(t)P^{-1}(t)dt, x\in(x_{j-1},x_j),\\ \int\limits_{x}^{x_{j+1}}\Phi_j^{(2n)}(t)P^{-1}(t)dt, x\in(x_j,x_{j+1}),\\ 0, x\notin[x_{j-1},x_{j+1}],\\ j=\overline{1,N-1}.\end{cases}\qquad \Phi_j^{(2n+2)}(x)=\begin{cases}\int\limits_{x_{j-1}}^{x}\Phi_j^{(2n+1)}(t)Q(t)dt, x\in(x_{j-1},x_j),\\ \int\limits_{x}^{x_{j+1}}\Phi_j^{(2n+1)}(t)Q(t)dt, x\in(x_j,x_{j+1}),\\ 0, x\notin[x_{j-1},x_{j+1}],\\ j=\overline{1,N-1}.\end{cases}$$

Further, dealing as at the derivation of (12), (13) we find

$$U_{j+1}=\left(E_m+\int\limits_{x_j}^{x_{j+1}}\sum_{k=1}^{n-1}\Phi_j^{(2k-1)}(x)Q(x)dx\right)U_j+\sum_{k=1}^{n}\Phi_j^{(2k-1)}(x_j+0)W_j$$
$$-\int\limits_{x_j}^{x_{j+1}}\sum_{k=1}^{n}\Phi_j^{(2k-1)}(x)F(x)dx+\int\limits_{x_j}^{x_{j+1}}\Phi_j^{(2n-1)}(x)Q(x)U(x)dx; \tag{15}$$

$$U_{j-1}=\left(E_m+\int\limits_{x_{j-1}}^{x_j}\sum_{k=1}^{n-1}\Phi_j^{(2k-1)}(x)Q(x)dx\right)U_j-\sum_{k=1}^{n}\Phi_j^{(2k-1)}(x_j-0)W_j$$
$$-\int\limits_{x_{j-1}}^{x_j}\sum_{k=1}^{n}\Phi_j^{(2k-1)}(x)F(x)dx+\int\limits_{x_{j-1}}^{x_j}\Phi_j^{(2n-1)}(x)Q(x)U(x)dx; \tag{16}$$

$$U_{j+1}=\left(E_m+\int\limits_{x_j}^{x_{j+1}}\sum_{k=1}^{n}\Phi_j^{(2k-1)}(x)Q(x)dx\right)U_j+\sum_{k=1}^{n}\Phi_j^{(2k-1)}(x_j+0)W_j$$
$$-\int\limits_{x_j}^{x_{j+1}}\sum_{k=1}^{n}\Phi_j^{(2k-1)}(x)F(x)dx+\int\limits_{x_j}^{x_{j+1}}\Phi_j^{(2n)}(x)P^{-1}(x)W(x)dx; \tag{17}$$

$$U_{j-1}=\left(E_m+\int\limits_{x_{j-1}}^{x_j}\sum_{k=1}^{n}\Phi_j^{(2k-1)}(x)Q(x)dx\right)U_j-\sum_{k=1}^{n}\Phi_j^{(2k-1)}(x_j-0)W_j$$
$$-\int\limits_{x_{j-1}}^{x_j}\sum_{k=1}^{n}\Phi_j^{(2k-1)}(x)F(x)dx-\int\limits_{x_{j-1}}^{x_j}\Phi_j^{(2n)}(x)P^{-1}(x)W(x)dx. \tag{18}$$

Let us introduce

$$\psi_j^{(n)}(x)=\sum_{k=1}^{n}\Phi_j^{(2k-1)}(x); n=1,2,\dots \tag{19}$$

Then, again using $[W]_{x_j}=0$, we eliminate W_j in (23),(24) to get:

$$\frac{U_j-U_{j+1}}{\psi_j^{(n)}(x_j+0)}+\frac{U_j-U_{j-1}}{\psi_j^{(n)}(x_j-0)}+\left(\int\limits_{x_{j-1}}^{x_j}\frac{\psi_j^{(n)}(x)}{\psi_j^{(n)}(x_j-0)}Q(x)dx+\int\limits_{x_j}^{x_{j+1}}\frac{\psi_j^{(n)}(x)}{\psi_j^{(n)}(x_j+0)}Q(x)dx\right)U_j$$
$$-\int\limits_{x_{j-1}}^{x_j}\frac{\Phi_j^{(2n)}(x)P^{-1}(x)}{\psi_j^{(n)}(x_j-0)}W(x)dx+\int\limits_{x_j}^{x_{j+1}}\frac{\Phi_j^{(2n)}(x)P^{-1}(x)}{\psi_j^{(n)}(x_j+0)}W(x)dx \tag{20}$$
$$=\int\limits_{x_{j-1}}^{x_j}\frac{\psi_j^{(n)}(x)}{\psi_j^{(n)}(x_j-0)}F(x)dx+\int\limits_{x_j}^{x_{j+1}}\frac{\psi_j^{(n)}(x)}{\psi_j^{(n)}(x_j+0)}F(x)dx.$$

In a similar way, from (15), (16) we find:

$$\frac{U_j-U_{j+1}}{\psi_j^{(n)}(x_j+0)}+\frac{U_j-U_{j-1}}{\psi_j^{(n)}(x_j-0)}+\left(\int\limits_{x_{j-1}}^{x_j}\frac{\psi_j^{(n-1)}(x)}{\psi_j^{(n)}(x_j-0)}Q(x)dx+\int\limits_{x_j}^{x_{j+1}}\frac{\psi_j^{(n-1)}(x)}{\psi_j^{(n)}(x_j+0)}Q(x)dx\right)U_j$$

$$+\int\limits_{x_{j-1}}^{x_j}\frac{\Phi_j^{(2n-1)}(x)}{\psi_j^{(n)}(x_j-0)}Q(x)U(x)dx+\int\limits_{x_j}^{x_{j+1}}\frac{\Phi_j^{(2n-1)}(x)}{\psi_j^{(n)}(x_j+0)}Q(x)U(x)dx \tag{21}$$

$$=\int\limits_{x_{j-1}}^{x_j}\frac{\psi_j^{(n)}(x)}{\psi_j^{(n)}(x_j-0)}F(x)dx+\int\limits_{x_j}^{x_{j+1}}\frac{\psi_j^{(n)}(x)}{\psi_j^{(n)}(x_j+0)}F(x)dx.$$

Note that the identities (9), (21) contain only the solution U, while (14), (20) - the solution U and the flux W.

The integral identities (20), (21) can be used for derivation of high-order difference schemes.The question is in the approximations of the integrals $\Phi_j^{(k)}(x)$.

Similar identities can be obtained for the flux W.

DERIVATION OF DIFFERENCE SCHEMES

Let introduce the operator

$$Q_\alpha^k(\Omega)\ni A^{(n)}V\to f\in Q_\alpha^{k-2}(\Omega),\ k\geq 2,\ k-integer,$$

$$\left(A^{(n)}V\right)_j=-\frac{1}{\hbar_j}\left[\frac{V_{j+1}-V_j}{\psi_j^{(n)}(x_{j+0})}+\frac{V_j-V_{j-1}}{\psi_j^{(n)}(x_{j-0})}\right]$$

$$+\frac{1}{\hbar_j}\left[\int_{x_{j-1}}^{x_j}\frac{\psi_j^{(n-1)}(x)}{\psi_j^{(n)}(x_{j-0})}Q(x)dx+\int_{x_j}^{x_{j+1}}\frac{\psi_j^{(n-1)}(x)}{\psi_j^{(n)}(x_{j+0})}Q(x)dx\right]V_j \tag{22}$$

$$+\frac{1}{\hbar_j}\left[\int_{x_{j-1}}^{x_j}\frac{\Phi_j^{(2n-1)}(x)Q(x)}{\psi_j^{(n)}(x_{j-0})}V(x)dx+\int_{x_j}^{x_{j+1}}\frac{\Phi_j^{(2n-1)}(x)Q(x)}{\psi_j^{(n)}(x_{j+0})}V(x)dx\right],$$

$$\left(f^{(n)}\right)_j=\frac{1}{\hbar_j}\left[\int_{x_{j-1}}^{x_j}\frac{\psi_j^{(n)}(x)}{\psi_j^{(n)}(x_{j-0})}F(x)dx-\int_{x_j}^{x_{j+1}}\frac{\psi_j^{(n)}(x)}{\psi_j^{(n)}(x_{j+0})}F(x)dx\right]. \tag{23}$$

Now the identity (20) reads as follows:

$$A^{(n)}V=F^{(n)}. \tag{24}$$

The substitution $V(x):=V(x_j)$ in the last two integrals in (22) leads to three-point difference schemes:

$$-a_j^{(n)}V_{j-1}+c_j^{(n)}V_j-b_j^{(n)}V_{j+1}=f_j^{(n)},\ j=1,\dots,N-1, \tag{25}$$

$$V_0=U_0,\ V_N=U_1,\ n=1,2,\dots$$

$$a_j^{(n)}=\frac{1}{\hbar_j\psi_j^{(n)}(x_j-0)},\ b_j^{(n)}=\frac{1}{\hbar_j\psi_j^{(n)}(x_j+0)}, \tag{26}$$

$$c_j^{(n)}=a_j^{(n)}+b_j^{(n)}+\frac{1}{\hbar_j}\left[\int\limits_{x_{j-1}}^{x_j}\frac{\psi_j^{(n)}(x)}{\psi_j^{(n)}(x_j-0)}Q(x)dx+\int\limits_{x_j}^{x_{j+1}}\frac{\psi_j^{(n)}(x)}{\psi_j^{(n)}(x_j+0)}Q(x)dx\right], \tag{27}$$

$$f_j^{(n)}=\frac{1}{\hbar_j}\left[\int\limits_{x_{j-1}}^{x_j}\frac{\psi_j^{(n)}(x)}{\psi_j^{(n)}(x_j-0)}F(x)dx+\int\limits_{x_j}^{x_{j+1}}\frac{\psi_j^{(n)}(x)}{\psi_j^{(n)}(x_j+0)}F(x)dx\right]. \tag{28}$$

The scheme (25) is applicable to practical computations only for such functions P,Q,F, when the integrals

$$\Phi_j^{(2n-1)}(x_j \pm 0), \Phi_j^{(2n)}(x_j \pm 0), \; j = 0,\ldots,N, \; n = 1,2,\ldots$$

could be explicitly calculated. Therefore, we need effective numerical integration.
In order to be derived from (25)-(28) finite difference schemes up to fourth order classical formulas (midpoint rule, trapezoid rule, Simpson's rule, etc.) can be used. For example, to obtain the popular "standard interface formula" we apply the midpoint rule to $a_j^{(1)}, b_j^{(1)}$ and the trapezoid rule to the other integrals in $c_j^{(1)}$ and $f_j^{(1)}$:

$$a_j^{(1)} = \frac{P_{j-1/2}}{h_j \hbar_j}, b_j^{(1)} = \frac{P_{j+1/2}}{h_{j+1}\hbar_j}, c_j^{(1)} = a_j^{(1)} + b_j^{(1)} + \frac{h_j Q_j^- + h_{j+1} Q_j^+}{h_j + h_{j+1}}, \; f_j^{(1)} = \frac{h_j F_j^- + h_{j+1} F_j^+}{h_j + h_{j+1}}.$$

To obtain the fourth order formula we apply the Simpson's rule to $a_j^{(2)}, b_j^{(2)}, c_j^{(2)}, f_j^{(2)}$ as follows.

$$a_j^{(2)} = \frac{1}{\hbar_j \psi_j^{(2)}(x_j - 0)}, \quad b_j^{(2)} = \frac{1}{\hbar_j \psi_j^{(2)}(x_j + 0)}, \quad \psi_j^{(2)}(x_j \pm 0) = \Phi_j^{(1)}(x_j \pm 0) + \Phi_j^{(3)}(x_j \pm 0),$$

$$\Phi_j^{(1)}(x_j - 0) = \frac{h_j}{6}\left(P^{-1}(x_{j-1}) + 4P^{-1}(x_{j-1/2}) + P^{-1}(x_j)\right) + O(h_j^5),$$

$$\Phi_j^{(1)}(x_j + 0) = \frac{h_{j+1}}{6}\left(P^{-1}(x_{j+1}) + 4P^{-1}(x_{j+1/2}) + P^{-1}(x_j)\right) + O(h_{j+1}^5),$$

$$\Phi_j^{(3)}(x_j - 0) = \frac{h_j}{6}\Phi_j^{(2)}(x_j - 0)P^{-1}(x_j) + \frac{2}{3}h_j \Phi_j^{(2)}(x_{j-1/2})P^{-1}(x_{j-1/2}) + O(h_j^5),$$

$$\Phi_j^{(3)}(x_j + 0) = \frac{h_{j+1}}{6}\Phi_j^{(2)}(x_j + 0)P^{-1}(x_j) + \frac{2}{3}h_{j+1} \Phi_j^{(2)}(x_{j+1/2})P^{-1}(x_{j+1/2}) + O(h_{j+1}^5),$$

$$\Phi_j^{(2)}(x) = \begin{cases} \dfrac{(x - x_{j-1})^2}{2} P^{-1}\left(\dfrac{x_{j-1} + 2x}{3}\right) Q\left(\dfrac{2x_{j-1} + x}{3}\right) + O(h_j^4), & x \in (x_{j-1}, x_j), \\ \dfrac{(x_{j+1} - x)^2}{2} P^{-1}\left(\dfrac{x_{j+1} + 2x}{3}\right) Q\left(\dfrac{2x_{j+1} + x}{3}\right) + O(h_{j+1}^4), & x \in (x_j, x_{j+1}), \\ 0, \quad x \notin [x_{j-1}, x_{j+1}], \\ j = \overline{1, N-1}. \end{cases}$$

$$c_j^{(2)} = a_j^{(2)} + b_j^{(2)} + \frac{2h_j}{3\hbar_j}\frac{\Phi_j^{(2)}(x_{j-1/2})Q(x_{j-1/2})}{\Phi_j^{(1)}(x_j - 0)} + \frac{1}{3}Q(x_j) + \frac{2h_{j+1}}{3\hbar_j}\frac{\Phi_j^{(2)}(x_{j+1/2})Q(x_{j-1/2})}{\Phi_j^{(1)}(x_j + 0)} + O(\hbar_j^4)$$

$$f_j^{(2)} = \frac{2h_j}{3\hbar_j}\frac{\Phi_j^{(2)}(x_{j-1/2})F(x_{j-1/2})}{\Phi_j^{(1)}(x_j - 0)} + \frac{1}{3}F(x_j) + \frac{2h_{j+1}}{3\hbar_j}\frac{\Phi_j^{(2)}(x_{j+1/2})F(x_{j-1/2})}{\Phi_j^{(1)}(x_j + 0)} + O(\hbar_j^4).$$

For $\Phi_j^{(2)}(x)$ we compute the surface of the rectangular triangle and multiplied by the values of the functions at the median center, which gives fourth order accuracy of the double integral (see [2]).

CONVERGENCE

Before to formulate the main theorem we will prove the assertion.
Lemma The coefficients of scheme (25) satisfy

$$\hbar_{j+1} a_{j+1}^{(n)} = \hbar_j b_j^{(n)}, \; j = 1,\ldots,N-1, \; n = 1,2,\ldots$$

Proof It follows from (26) that we must prove

$$\psi_{j+1}^{(n)}(x_{j+1} - 0) = \psi_j^{(n)}(x_j + 0), \; j = 1,\ldots,N-1, \; n = 1,2,\ldots$$

But, in view of formula (19) for $\psi_j^{(n)}(x)$, the last equalities are equivalent to the following ones

$$\Phi_{j+1}^{(2k-1)}(x_{j+1} - 0) = \Phi_j^{(2k-1)}(x_j + 0) \; for \; k = 1,\ldots,n.$$

This can be easily checked by change of the order of integration in the corresponding integrals.

Theorem Let $P \in Q_\alpha^1,\ Q, F \in Q_\alpha^0$,while U be the solution of the differential problem (1), (2) and U^h - the solution of the difference problem (25)-(28). Then

$$\left\| U - U^h \right\|_\infty \le C\bar{h}^{2n},$$

where $\bar{h} = \sqrt[2n]{\sum_{j=1}^{N} h_j^{2n+1}}$ and the constant C doesn't depend on $\bar{h}$.

The proof is similar to this of Theorem 4.1 in [1].

NUMERICAL EXPERIMENTS

In this section are presented samples of numerical experiments illustrating the accuracy of the schemes derived in Section 3.

Example 1. In this example the coefficients are:

$P(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $Q(x) = \begin{pmatrix} 2(x+1)^2 & -(x^3+1) \\ -2\cos(\pi x/4) & 2.2e^{-x+1} \end{pmatrix}$, $F(x)$ - the exact solution to this system is: $U(x) = \left(x^2(1-x^2), \sin(\pi x)\right)^T$.

Example 2. In this example the coefficients are:

$P(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $Q(x) = \begin{pmatrix} 7+x & -(1+x^2) & -2 \\ -1 & 6+x^2 & -(2+x) \\ -2 & -(1+e^x) & 5+x^3 \end{pmatrix}$, $F(x)$ - the exact solution to this system is: $U(x) = \left(x^2(1-x^2), \sin(\pi x), x^3(1-x)\right)^T$.

$$\left\| z_h^{(n)} \right\|_\infty = \max_{0 \le j \le N} | z^{(n)}(x_j) |, \quad h = 1/N,$$

of the discretization error is tabulated together with an approximate rate of convergence

$$p_n = \log_2 \frac{\left\| z_{2h}^{(n)} \right\|_\infty}{\left\| z_h^{(n)} \right\|_\infty}.$$

Table 1: $\mathit{Example}\ 1: \mathit{the\ solution}\ \ u_1$

N	4	8	16	32	64	128	256
$\left\| z_h^{(2)} \right\|$	7.331e-3	1.890e-3	4.779e-4	1.199e-4	2.998e-5	7.498e-6	1.874e-6
p_2	1.9559	1.9833	1.9954	1.9991	1.9996	1.9999	
$\left\| z_h^{(4)} \right\|$	2.175e-4	1.325e-5	8.217e-7	8.217e-7	3.205e-9	1.911e-10	1.211e-11
p_4	4.0365	4.0116	4.0031	3.9991	4.0682	3.9797	

Table 2: $\mathit{Example}\ 1: \mathit{the\ solution}\ \ u_2$

N	4	8	16	32	64	128	256
$\left\| z_h^{(2)} \right\|$	3.705e-2	9.138e-3	2.277e-3	5.688e-4	1.422e-4	3.555e-5	8.889e-6
p_2	2.0196	2.0049	2.0012	1.9998	1.9999	2.0000	
$\left\| z_h^{(4)} \right\|$	5.904e-4	3.954e-5	2.475e-6	1.547e-7	9.679e-9	6.059e-10	3.868e-11
p_4	3.9004	3.9979	3.9995	3.9988	3.9976	3.9694	

Table 3: *Example* 2 : *the solution U*

N	4	8	16	32	64	128	256
$\left\|z_h^{(4)}\right\|$	1.209e-4	7.590e-6	4.727e-7	2.834e-8	1.637e-9	8.768e-11	3.652e-12
u_1	3.9932	4.0050	4.0602	4.1135	4.2228	4.5854	
$\left\|z_h^{(4)}\right\|$	9.232e-4	5.856e-5	3.749e-6	2.447e-7	1.647e-8	1.172e-9	8.513e-11
u_2	3.9786	3.9653	3.9378	3.8929	3.8121	3.7838	
$\left\|z_h^{(4)}\right\|$	3.029e-4	2.116e-5	1.315e-6	8.038e-8	4.825e-9	2.797e-10	1.522e-11
u_3	3.8391	4.0083	4.0321	4.0583	4.1085	4.2000	

The approximations were computed on uniform meshes with $N = 4,8,16,\ldots,256$. The strong norm in Tables subsctipt ∞ omitted.

Example 3. In this example the coefficients are: $P(x) = \begin{pmatrix} \varepsilon^2 & 0 \\ 0 & \varepsilon^2 \end{pmatrix}$, $Q(x) = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$, $F(x) = (2,3)^T$.

The exact solution to this system can be obtained, so that we will be able to calculate the exact numerical errors.

Example 3 is solved on Shishkin mesh (see [5]) with $\sigma = 3$, $\tau = \min\{1/4, \sigma\varepsilon \ln N/\alpha\}$.

Table 4: *Example* 3 : *the solution* u_1

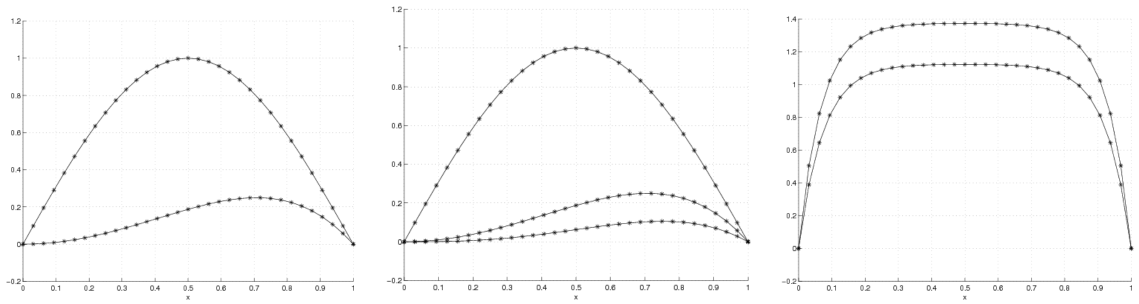
ε/N	16	32	64	128	256	512	1024
1	1.528e-8	9.545e-10	5.965e-11	3.727e-12	3.100e-13		
p_4	4.0005	4.0000	4.0007	3.5875			
10^{-1}	2.237e-4	1.546e-5	9.779e-7	6.132e-8	3.839e-9	2.410e-10	1.574e-11
p_4	3.8543	3.9832	3.9951	3.9977	3.9934	3.9367	
10^{-2}	4.145e-3	1.014e-3	1.401e-4	1.729e-5	1.871e-6	1.879e-7	1.793e-8
p_4	2.0310	2.8554	3.0185	3.2082	3.3160	3.3896	
10^{-3}	4.144e-3	1.014e-3	1.401e-4	1.729e-5	1.871e-6	1.879e-7	1.793e-8
p_4	2.0309	2.8554	3.0185	3.2082	3.3160	3.3896	
10^{-4}	4.144e-3	1.014e-3	1.401e-4	1.729e-5	1.871e-6	1.879e-7	1.793e-8
p_4	2.0309	2.8554	3.0185	3.2082	3.3160	3.3896	
10^{-5}	4.144e-3	1.014e-3	1.401e-4	1.729e-5	1.871e-6	1.879e-7	1.793e-8
p_4	2.0309	2.8554	3.0185	3.2082	3.3160	3.3896	
10^{-6}	4.144e-3	1.014e-3	1.401eE-4	1.729e-5	1.871e-6	1.879e-7	1.793e-8
p_4	2.0309	2.8554	3.0185	3.2082	3.3160	3.3896	
$\left(\frac{\ln N}{N}\right)^4$	3.208E-5	4.895E-6	6.343E-7	7.345E-8	7.831E-9	7.840E-10	7.469E-11
p_4	2.7123	2.9479	3.1104	3.2294	3.3203	3.3920	

Table 5: *Example* 3 : *the solution* u_2

ε/N	16	32	64	128	256	512	1024
1	5.702e-8	3.568e-9	2.231e-10	1.401e-11	9.251e-13		
p_4	3.9983	3.9994	3.9934	3.9204			
10^{-1}	4.682e-4	3.246e-5	2.085e-6	1.312e-7	8.217e-9	5.140e-10	3.259e-11
p_4	3.8503	3.9605	3.9900	3.9975	3.9986	3.9792	

10^{-2}	5.801e-3	1.768e-3	2.894e-4	3.635e-5	3.943e-6	4.017e-7	3.838e-8
p_4	1.7143	2.6108	2.9928	3.2049	3.2949	3.3876	
10^{-3}	5.800e-3	1.768e-3	2.894e-4	3.635e-5	3.943e-6	4.017e-7	3.838e-8
p_4	1.7142	2.6108	2.9928	3.2049	3.2949	3.3876	
10^{-4}	5.800e-3	1.768e-3	2.894e-4	3.635e-5	3.943e-6	4.017e-7	3.838e-8
p_4	1.7142	2.6108	2.9928	3.2049	3.2949	3.3876	
10^{-5}	5.800e-3	1.768e-3	2.894e-4	3.635e-5	3.943e-6	4.017e-7	3.838e-8
p_4	1.7142	2.6108	2.9928	3.2049	3.2949	3.3876	
10^{-6}	5.800e-3	1.768e-3	2.894e-4	3.635e-5	3.943e-6	4.017e-7	3.838e-8
p_4	1.7142	2.6108	2.9928	3.2049	3.2949	3.3876	

The computations were done with MATLAB. For the schemes of order 2, 4 the results are contained in Tables in which the convergence rates are clearly observed.



1) 2) 3)

Figure 1: Approximate and exact solution for 1) Example 1, 2) Example 2, 3) Example 3, for $N = 32$ and $\varepsilon = 10^{-1}$.

CONCLUSIONS

In this paper we generalized the method of integral identities for one-dimensional scalar interface problems in order to obtain conservative high-order difference schemes for systems of ODEs. This approach can be applied for construction of high-order approximations to parabolic and elliptic degenerate equations. Our schemes, in the case $P(x) := \varepsilon^2 P(x)$ have uniform convergence with respect to the small parameter ε and easily can be implementated to singularly perturbed problems.

ACKNOWLEDGEMENTS

This work was supported by the project: 2017-FPNO-03.

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ДИФЕРЕНЧНИ СХЕМИ ОТ ВИСОК РЕД НА ТОЧНОСТ НА ОСНОВАТА НА ИНТЕГРАЛНИ ТЪЖДЕСТВА ЗА СЛАБО СВЪРЗАНИ СИСТЕМИ ОБИКНОВЕНИ ДИФЕРЕНЦИАЛНИ УРАВНЕНИЯ

Иванка Ангелова

Русенски университет "Ангел Кънчев"

Резюме: *В статията се разглеждат диференчни схеми от висок ред на точност за системи от диференциални уравнения от тип реакция-дифузия. Формулите се основават на интегрални тъждества, които са обобщение на интегралните тъждества от типа на Марчук за диференциални уравнения от втори ред. От тях се извеждат диференчни схеми от ред* $O(h^2), O(h^4),\ldots$ *За численото интегриране са използвани познати квадратурни формули за еднократни интеграли и аналог на формулата на средната точка за двукратен интеграл, изведена в [2]. Формулирана е теорема за сходимост. Числените експерименти потвърждават теоретичните резултати.*

Ключови думи: *Системи диференциални уравнения от тип реакция-дифузия, интегрални тъждества, диференчнаи схеми.*

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