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Union of Scientists - Ruse
16, Konstantin Irechek Street
7000 Ruse
BULGARIA
Phone: (++359 82) 828 135,
(++359 82) 841 634
E-mail: suruse@uni-ruse.bg
web: suruse.uni-ruse.bg

Contacts with Editor
Phone: (++359 82) 888 738
E-mail: zzdravkova@uni-ruse.bg

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IMMERSED INTERFACE FINITE ELEMENT METHOD FOR DIFFUSION PROBLEM WITH LOCALIZED TERMS

Ivan Georgiev, Juri Kandilarov

Angel Kanchev University of Ruse

***Abstract**: In this paper we consider the diffusion problem with local own source. A weak formulation of it is done and then the immersed interface finite element method (IIFEM) is applied for the numerical solution. For discretization in time it is used Rothe's method with weights, and then the special basis functions, which fulfill the jump conditions on the interface, are introduced. Numerical results, confirming second order of accuracy in maximum norm, are presented.*

***Keywords:** immersed interface, finite element method, diffusion equation, local sources, Rothe's method*

INTRODUCTION

Let us consider the following parabolic problem with discontinuous coefficients and local own source:

$$u_t(x,t)-(\beta u_x(x,t))_x+q(x,t)u(x,t)=f(x,t)-g(u(x,t))\delta(x-\zeta) \qquad (x,t)\in Q_T=\Omega\times(0,1] \tag{1}$$

with initial and boundary conditions

$$\begin{aligned} &u(x,0)=u_0(x),\\ &u(0,t)=u_L(t),\, u(1,t)=u_R(t). \end{aligned} \tag{2}$$

Here $\Omega=(0,1)$, the functions q,f and g are continuous in $\overline{Q_T}\setminus\Gamma_T$, $\Gamma_T=\{\zeta\}\times[0,1]$, δ is the Dirac-delta function, $0<\zeta<1$ and β is piecewise continuous of the form

$$\beta(x)=\begin{cases}\beta^-(x), & 0\le x\le\zeta,\\ \beta^+(x), & \zeta\le x\le 1.\end{cases}$$

Under some assumption of smoothness of the solution the problem (1)-(2) can be rewritten in the classical way [9] as follows:

$$u_t(x,t)-(\beta u_x(x,t))_x+q(x,t)u(x,t)=f(x,t) \qquad (x,t)\in Q_T=\Omega\times(0,1]) \tag{3}$$

with initial and boundary conditions (2) and jump conditions on the interface Γ_T

$$\begin{aligned} &[u(x,t)]_{x=\zeta}=u(\zeta^+,t)-u(\zeta^-,t)=0\\ &[\beta u_x(x,t)]_{x=\zeta}=\beta^+u_x(\zeta^+,t)-\beta^-u_x(\zeta^-,t)=g(u(\zeta,t)). \end{aligned} \tag{4}$$

Problems of this type arise in chemical processes with local reactions on a catalytic surface [1], where the global solvability and blow-up of the solution in finite time are also studied. Compatibility conditions for the regularity of the solution are described in [3]. Exis tence and uniqueness of the variety of elliptic and parabolic interface problems are discussed in the book [9]. Special difference schemes for such problems are considered [4].

In this paper the IIFEM is applied for the proposed problem. IIFEM for interface problems with homogeneous interface conditions, or jump conditions, that are known functions, have been studied in [5-7]. In our problem the jump of the flux at the interface depends on the unknown solution. For elliptic problems of this type an IIFEM is studied in [2].

The paper is organized as follows. In the next section the weak formulation of the problem is given. Then using Rothe's method combined with the method of weights the semi-discretization in time is done. IIFEM is applied for approximation in space. Numerical experiments are presented in the last section.

WEAK FORMULATION OF THE PROBLEM

Let for simplicity consider the case of linear own source, i.e. $g(u(x,t)) = K(t)u(x,t)$, where $K(t) > 0,\ \forall t \in [0,1]$ is a continuous function.

Let introduce the usual Sobolev space $H^1(0,1)$, the bilinear form

$$a(u,v) = \int_0^1 (\beta(x)u_x'(x,t)v'(x) + q(x,t)u(x,t)v(x)dx + K(t)u(\zeta,t)v(\zeta,t) \qquad (5)$$

and linear form

$$b(f,v) = \int_0^1 f(x,t)v(x)dx. \qquad (6)$$

With $(u_t(x,t),v) = \int_0^1 u_t(x,t)v(x)dx$ we denote the inner product in the space $L^2(0,1)$.

Then, the weak solution of the problem (3), (4) is the function $u \in H^1(0,1), \forall t \in (0,1]$, such that

$$(u_t,v) + a(u,v) = b(f,v) \qquad \forall v \in H^1(0,1), \qquad (7)$$

and $u(x,t)$ satisfies the conditions in (2).

Using energetic method, see Chapter 1 of [9], it has been proved that if $u_0(x),\ q,\ f \in L^2(0,1),$ then the solution of (3)-(4) is unique $u \in H^1(0,1), \forall t \in (0,1]$ satisfying the weak problem (7).

METHOD OF WEIGHTS

Let introduce an uniform mesh in the time $t \in [0,1]$ with constant time step $\tau = 1/M,$ $t_m = m\tau,\ m = 0,...,M$, where *M* is a positive integer. Let also with $z_m(x)$ denote the numerical approximation of $u(x,t_m)$ on the m-th time layer $m = 1,...,M,$ and σ, $\sigma \in [0,1]$ is a weight. Then the semi-discretization of the problem in time [8] looks as follows:

$$\frac{z_m(x)-z_{m-1}(x)}{\tau}-(\sigma\beta z_m'(x))'-((1-\sigma)\beta z_{m-1}'(x))'+\sigma q(x,t_m)z_m(x)+(1-\sigma)q(x,t_{m-1})z_{m-1}(x)$$
$$=\sigma f(x,t_m)+(1-\sigma)f(x,t_{m-1})-\sigma\delta(x-\zeta)K(t_m)z_m(x)-(1-\sigma)\delta(x-\zeta)K(t_{m-1})z_{m-1}(x),\quad x\in(0,1) \tag{8}$$

for $m=1,...,M,$ initial and boundary conditions

$$\begin{aligned} u(x,0)&=z_0(x)=u_0(x),\\ u(0,t_m)&=z_m(0)=u_L(t_m),\\ u(1,t_m)&=z_m(1)=u_R(t_m), \end{aligned} \tag{9}$$

and jump conditions on the interface

$$\begin{aligned} [z_m(x)]_{x=\zeta}&=z_m(\zeta^+)-z_m(\zeta^-)=0,\\ [\beta z'_m(x)]_{x=\zeta}&=\beta^+\frac{\partial z_m}{\partial x}(\zeta^+)-\beta^-\frac{\partial z_m}{\partial x}(\zeta^-)=K(t_m)z_m(\zeta). \end{aligned} \tag{10}$$

IMMERSED INTERFACE FINITE ELEMENT METHOD

Next we introduce uniform mesh in space direction x, $x_i=ih,\ i=0,...,N,$ with $h=1/N.$ Let J be the number, for which $x_J\le\zeta<x_{J+1}$. Let c_i^m are the unknown coefficients at x_i and t_m. Then from the idea of the FEM the numerical solution on every time layer m is a linear combination $z_m^h=\sum_{i=0}^{N}c_i^m\phi_i(x)$ of standard basic functions ϕ_i, $i\ne J,\ i\ne J+1$:

$$\phi_i(x)=\begin{cases}\dfrac{x-x_{i-1}}{h}, & x_{i-1}\le x<x_i\\ \dfrac{x_{i+1}-x}{h}, & x_i\le x\le x_{i+1}\\ 0, & elsewhere\end{cases},$$

and two modified basis functions ϕ_J and ϕ_{J+1}:

$$\phi_J(x)=\begin{cases}0, & 0\le x<x_{J-1}\\ \dfrac{x-x_{J-1}}{h}, & x_{J-1}\le x<x_J\\ \alpha_{11}x+\beta_{11}, & x_J\le x<\zeta\\ \alpha_{12}x+\beta_{12} & \zeta\le x<x_{J+1}\\ 0 & x_{J+1}\le x\le 1\end{cases}\qquad \phi_{J+1}(x)=\begin{cases}0, & 0\le x<x_j\\ \alpha_{21}x+\beta_{21}, & x_J\le x<\zeta\\ \alpha_{22}x+\beta_{22}, & \zeta\le x<x_{J+1}\\ \dfrac{x_{J+2}-x}{h}, & x_{J+1}\le x<x_{J+2}\\ 0 & x_{J+2}\le x\le 1\end{cases}$$

The coefficients α_{ij} and β_{ij}, $i,j=1,2$ must satisfy the following systems of linear algebraic equations (SLAE)

$$\begin{vmatrix}\alpha_{11}x_J+\beta_{11}=1\\ \alpha_{12}x_{J+1}+\beta_{12}=0\\ \alpha_{12}\zeta+\beta_{12}-\alpha_{11}\zeta-\beta_{11}=0\\ \alpha_{12}-\alpha_{11}=K(\alpha_{11}\zeta+\beta_{11})\end{vmatrix} \quad \text{and} \quad \begin{vmatrix}\alpha_{21}x_J+\beta_{21}=0\\ \alpha_{22}x_{J+1}+\beta_{22}=1\\ \alpha_{22}\zeta+\beta_{22}-\alpha_{21}\zeta-\beta_{21}=0\\ \alpha_{22}-\alpha_{21}=K(\alpha_{21}\zeta+\beta_{21})\end{vmatrix}.$$

Its solutions are:

$$\alpha_{11}=-\frac{1+K\rho_{I+1}}{h+K\rho_I\rho_{I+1}},\ \beta_{11}=\frac{K\zeta\rho_{I+1}+x_{I+1}}{h+K\rho_I\rho_{I+1}},\ \alpha_{12}=-\frac{1}{h+K\rho_I\rho_{I+1}},\ \beta_{12}=\frac{x_{J+1}}{h+K\rho_I\rho_{I+1}}$$

$$\alpha_{21}=\frac{1}{h+K\rho_I\rho_{I+1}},\ \beta_{21}=-\frac{x_J}{h+K\rho_I\rho_{I+1}},\ \alpha_{22}=\frac{1+K\rho_I}{h+K\rho_I\rho_{I+1}},\ \beta_{22}=-\frac{K\zeta\rho_I+x_J}{h+K\rho_I\rho_{I+1}}$$

$$\rho_I=\zeta-x_J\ \ \rho_{I+1}=x_{J+1}-\zeta\ .$$

In what follows, instead of $(.,.)$ for the scalar product we will use the notation $\langle .,. \rangle$, i.e. $<u,v>=\int_0^1 uvdx$. Then we multiply the semi-discrete equation (8) by the test function $v\in H^1(0,1)$, and integrating by parts with respect to x we get for $j=1,...,N$:

$$\begin{aligned}&<\sum_{i=0}^{N}\sigma\beta c_i^m\phi'^m_i,\phi'^m_j>+<\sum_{i=0}^{N}(\frac{1}{\tau}+\sigma q)c_i^m\phi_i^m,\phi_j^m>+(\sum_{i=0}^{N}(\sigma K(t_m)c_i^m\phi_i^m(\zeta))\phi_j^m(\zeta)=\\&<\sum_{i=0}^{N}\frac{1}{\tau}c_i^{m-1}\phi_i^{m-1},\phi_j^m>-<\sum_{i=0}^{N}(1-\sigma)\beta c_i^{m-1}\phi'^{m-1}_i,\phi'^m_j>+<\sum_{i=0}^{N}(1-\sigma)q\beta c_i^{m-1}\phi_i^{m-1},\phi_j^m>+\\&<\sigma f(x,t_m),\phi_j^m>+<(1-\sigma)f(x,t_{m-1}),\phi_j^m>-(\sum_{i=0}^{N}(1-\sigma)K(t_{m-1})c_i^{m-1}\phi_i^{m-1}(\zeta))\phi_j^m(\zeta).\end{aligned}\tag{11}$$

This SLAE can be presented in the matrix form as

$$(A^1+A^2+A^3)C^m=(B^1+B^2+B^3+B^4)C^{m-1}+(D^1+D^2),$$

where $C^{m-1}=(c_0^{m-1},c_1^{m-1},...,c_N^{m-1})^T$ and $C^m=(c_0^m,c_1^m,...,c_N^m)^T$ are vectors of the solution on the two consecutive time layers, with $C^0=(c_0^0,c_1^0,...,c_N^0)^T, c_i^0=u_0(x_i), c_0^m=u(0,t_m), c_N^m=u(1,t_m)$ - initial and boundary conditions. The elements of the other matrices are as follows:

$$A_{ij}^1=\begin{cases} 0 & i\neq j, i\neq j\pm 1 \\ \sigma\int_{x_{i-1}}^{x_{i+1}}\beta^-(x)\phi_i^m(x)'\phi_j^m(x)'dx & i=j\pm 1\, or\, i=j,\, and\, i<J \\ \sigma\int_{x_{i-1}}^{x_{i+1}}\beta^+(x)\phi_i^m(x)'\phi_j^m(x)'dx & i=j\pm 1\, or\, i=j,\, and\, i>J+1 \\ \sigma\int_{x_{J-1}}^{\zeta}\beta^-(x)\phi_J^m(x)'\phi_J^m(x)'dx+\sigma\int_{\zeta}^{x_{J+1}}\beta^+(x)\phi_J^m(x)'\phi_J^m(x)'dx & i=J,\ j=J \\ \sigma\int_{x_J}^{\zeta}\beta^-(x)\phi_J^m(x)'\phi_{J+1}^m(x)'dx+\sigma\int_{\zeta}^{x_{J+1}}\beta^+(x)\phi_J^m(x)'\phi_{J+1}^m(x)'dx & i=J,\ j=J+1 \\ \sigma\int_{x_J}^{\zeta}\beta^-(x)\phi_{J+1}^m(x)'\phi_J^m(x)'dx+\sigma\int_{\zeta}^{x_{J+1}}\beta^+(x)\phi_{J+1}^m(x)'\phi_J^m(x)'dx & i=J+1,\ j=J \\ \sigma\int_{x_J}^{\zeta}\beta^-(x)\phi_{J+1}^m(x)'\phi_{J+1}^m(x)'dx+\sigma\int_{\zeta}^{x_{J+2}}\beta^+(x)\phi_{J+1}^m(x)'\phi_{J+1}^m(x)'dx & i=J+1,\ j=J+1 \end{cases}$$

$$A_{ij}^2=\begin{cases} 0 & i\neq j, i\neq j\pm 1 \\ \int_{x_{i-1}}^{x_{i+1}}(\frac{1}{\tau}+\sigma q^-(x,t_m))\phi_i^m(x)\phi_j^m(x)dx & i=j\pm 1\, or\, i=j,\, and\, i<J \\ \int_{x_{i-1}}^{x_{i+1}}(\frac{1}{\tau}+\sigma q^+(x,t_m))\phi_i^m(x)\phi_j^m(x)dx & i=j\pm 1\, or\, i=j,\, and\, i>J+1 \\ \int_{x_{J-1}}^{\zeta}(\frac{1}{\tau}+\sigma q^-(x,t_m))\phi_J^m(x)\phi_J^m(x)dx+\int_{\zeta}^{x_{J+1}}(\frac{1}{\tau}+\sigma q^+(x,t_m))\phi_J^m(x)\phi_J^m(x)dx & i=J,\ j=J \\ \int_{x_J}^{\zeta}(\frac{1}{\tau}+\sigma q^-(x,t_m))\phi_J^m(x)\phi_{J+1}^m(x)dx+\int_{\zeta}^{x_{J+1}}(\frac{1}{\tau}+\sigma q^+(x,t_m))\phi_J^m(x)\phi_{J+1}^m(x)dx & i=J,\ j=J+1 \\ \int_{x_J}^{\zeta}(\frac{1}{\tau}+\sigma q^-(x,t_m))\phi_{J+1}^m(x)\phi_J^m(x)dx+\int_{\zeta}^{x_{J+1}}(\frac{1}{\tau}+\sigma q^+(x,t_m))\phi_{J+1}^m(x)\phi_J^m(x)dx & i=J+1,\ j=J \\ \int_{x_J}^{\zeta}(\frac{1}{\tau}+\sigma q^-(x,t_m))\phi_{J+1}^m(x)\phi_{J+1}^m(x)dx+\int_{\zeta}^{x_{J+2}}(\frac{1}{\tau}+\sigma q^+(x,t_m))\phi_{J+1}^m(x)\phi_{J+1}^m(x)dx & i=J+1,\ j=J+1 \end{cases}$$

$$A_{ij}^3=\begin{cases} \sigma K(t_m)\phi_J^m(\zeta)\phi_J^m(\zeta) & i=J,\ j=J \\ \sigma K(t_m)\phi_J^m(\zeta)\phi_{J+1}^m(\zeta) & i=J,\ j=J+1 \\ \sigma K(t_m)\phi_{J+1}^m(\zeta)\phi_J^m(\zeta) & i=J+1,\ j=J \\ \sigma K(t_m)\phi_{J+1}^m(\zeta)\phi_{J+1}^m(\zeta) & i=J+1,\ j=J+1 \\ 0 & other\, cases \end{cases};$$

$$B_{ij}^1=\begin{cases} 0 & i\neq j, i\neq j\pm 1\\ \int\limits_{x_{i-1}}^{x_{i+1}}(\frac{1}{\tau})\phi_i^{m-1}(x)\phi_j^m(x)dx & i=j\pm 1\, or\, i=j,\, and\, i<J\\ \int\limits_{x_{i-1}}^{x_{i+1}}(\frac{1}{\tau})\phi_i^{m-1}(x)\phi_j^m(x)dx & i=j\pm 1\, or\, i=j,\, and\, i>J+1\\ \int\limits_{x_{J-1}}^{\zeta}(\frac{1}{\tau})\phi_J^{m-1}(x)\phi_J^m(x)dx+\int\limits_{\zeta}^{x_{J+1}}(\frac{1}{\tau})\phi_J^{m-1}(x)\phi_J^m(x)dx & i=J,\ j=J\\ \int\limits_{x_J}^{\zeta}(\frac{1}{\tau})\phi_J^{m-1}(x)\phi_{J+1}^m(x)dx+\int\limits_{\zeta}^{x_{J+1}}(\frac{1}{\tau})\phi_J^{m-1}(x)\phi_{J+1}^m(x)dx & i=J,\ j=J+1\\ \int\limits_{x_J}^{\zeta}(\frac{1}{\tau})\phi_{J+1}^{m-1}(x)\phi^m{}_J(x)dx+\int\limits_{\zeta}^{x_{J+1}}(\frac{1}{\tau})\phi_{J+1}^{m-1}(x)\phi_J^m(x)dx & i=J+1,\ j=J\\ \int\limits_{x_J}^{\zeta}(\frac{1}{\tau})\phi_{J+1}^{m-1}(x)\phi_{J+1}^m(x)dx+\int\limits_{\zeta}^{x_{J+2}}(\frac{1}{\tau})\phi_{J+1}^{m-1}(x)\phi_{J+1}^m(x)dx & i=J+1,\ j=J+1 \end{cases}$$

$$B_{ij}^2=\begin{cases} 0 & i\neq j, i\neq j\pm 1\\ -(1-\sigma)\int\limits_{x_{i-1}}^{x_{i+1}}\beta^-(x)\phi_i^{m-1}(x)'\phi_j^m(x)'dx & i=j\pm 1\, or\, i=j,\, and\, i<J\\ -(1-\sigma)\int\limits_{x_{i-1}}^{x_{i+1}}\beta^+(x)\phi_i^{m-1}(x)'\phi_j^m(x)'dx & i=j\pm 1\, or\, i=j,\, and\, i>J+1\\ -(1-\sigma)\int\limits_{x_{J-1}}^{\zeta}\beta^-(x)\phi_J^{m-1}(x)'\phi_J^m(x)'dx-(1-\sigma)\int\limits_{\zeta}^{x_{J+1}}\beta^+(x)\phi_J^{m-1}(x)'\phi_J^m(x)'dx & i=J,\ j=J\\ -(1-\sigma)\int\limits_{x_J}^{\zeta}\beta^-(x)\phi_J^{m-1}(x)'\phi_{J+1}^m(x)'dx-(1-\sigma)\int\limits_{\zeta}^{x_{J+1}}\beta^+(x)\phi_J^{m-1}(x)'\phi_{J+1}^m(x)'dx & i=J,\ j=J+1\\ -(1-\sigma)\int\limits_{x_J}^{\zeta}\beta^-(x)\phi_{J+1}^{m-1}(x)'\phi_J^m(x)'dx-(1-\sigma)\int\limits_{\zeta}^{x_{J+1}}\beta^+(x)\phi_{J+1}^{m-1}(x)'\phi_J^m(x)'dx & i=J+1,\ j=J\\ -(1-\sigma)\int\limits_{x_J}^{\zeta}\beta^-(x)\phi_{J+1}^{m-1}(x)'\phi_{J+1}^m(x)'dx-(1-\sigma)\int\limits_{\zeta}^{x_{J+2}}\beta^+(x)\phi_{J+1}^{m-1}(x)'\phi_{J+1}^m(x)'dx & i=J+1,\ j=J+1 \end{cases}$$

$$B_{ij}^3=\begin{cases} 0 & i\neq j, i\neq j\pm 1\\ (1-\sigma)\int\limits_{x_{i-1}}^{x_{i+1}}q^-(x,t_{m-1})\phi_i^{m-1}(x)\phi_j^m(x)dx & i=j\pm 1\, or\, i=j,\, and\, i<J\\ (1-\sigma)\int\limits_{x_{i-1}}^{x_{i+1}}q^+(x,t_{m-1})\phi_i^{m-1}(x)\phi_j^m(x)dx & i=j\pm 1\, or\, i=j,\, and\, i>J+1\\ (1-\sigma)\int\limits_{x_{J-1}}^{\zeta}q^-(x,t_{m-1})\phi_J^{m-1}(x)\phi_J^m(x)dx+(1-\sigma)\int\limits_{\zeta}^{x_{J+1}}q^+(x,t_{m-1})\phi_J^{m-1}(x)\phi_J^m(x)dx & i=J,\ j=J\\ (1-\sigma)\int\limits_{x_J}^{\zeta}q^-(x,t_{m-1})\phi_J^{m-1}(x)\phi_{J+1}^m(x)dx+(1-\sigma)\int\limits_{\zeta}^{x_{J+1}}q^+(x,t_{m-1})\phi_J^{m-1}(x)\phi_{J+1}^m(x)dx & i=J,\ j=J+1\\ (1-\sigma)\int\limits_{x_J}^{\zeta}q^-(x,t_{m-1})\phi_{J+1}^{m-1}(x)\phi_J^m(x)dx+(1-\sigma)\int\limits_{\zeta}^{x_{J+1}}q^+(x,t_{m-1})\phi_{J+1}^{m-1}(x)\phi_J^m(x)dx & i=J+1,\ j=J\\ (1-\sigma)\int\limits_{x_J}^{\zeta}q^-(x,t_{m-1})\phi_{J+1}^{m-1}(x)\phi_{J+1}^m(x)dx+(1-\sigma)\int\limits_{\zeta}^{x_{J+2}}q^+(x,t_{m-1})\phi_{J+1}^{m-1}(x)\phi_{J+1}^m(x)dx & i=J+1,\ j=J+1 \end{cases}$$

$$B_{ij}^4=\begin{cases}-(1-\sigma)K(t_{m-1})\phi_J^{m-1}(\zeta)\phi_J^m(\zeta) & i=J,\ j=J\\ -(1-\sigma)K(t_{m-1})\phi_J^{m-1}(\zeta)\phi_{J+1}^m(\zeta) & i=J,\ j=J+1\\ -(1-\sigma)K(t_{m-1})\phi_{J+1}^{m-1}(\zeta)\phi_J^m(\zeta) & i=J+1,\ j=J\\ -(1-\sigma)K(t_{m-1})\phi_{J+1}^{m-1}(\zeta)\phi_{J+1}^m(\zeta) & i=J+1,\ j=J+1\\ 0 & other\quad cases\end{cases};$$

$$D_i^1=\begin{cases}\sigma\int_{x_{i-1}}^{x_{i+1}} f^-(x,t_m)\phi_i^m dx & i\neq J, i\neq J+1, i<J\\ \sigma\int_{x_{i-1}}^{x_{i+1}} f^+(x,t_m)\phi_i^m dx & i\neq J, i\neq J+1, i>J+1\\ \sigma\int_{x_{J-1}}^{\zeta} f^-(x,t_m)\phi_J^m dx+\sigma\int_{\zeta}^{x_{J+1}} f^+(x,t_m)\phi_J^m dx & i=J\\ \sigma\int_{x_J}^{\zeta} f^-(x,t_m)\phi_{J+1}^m dx+\sigma\int_{\zeta}^{x_{J+2}} f^+(x,t_m)\phi_{J+1}^m dx & i=J+1\end{cases}$$

$$D_i^2=\begin{cases}(1-\sigma)\int_{x_{i-1}}^{x_{i+1}} f^-(x,t_{m-1})\phi_i^m dx & i\neq J, i\neq J+1, i<J\\ (1-\sigma)\int_{x_{i-1}}^{x_{i+1}} f^+(x,t_{m-1})\phi_i^m dx & i\neq J, i\neq J+1, i>J+1\\ (1-\sigma)\int_{x_{J-1}}^{\zeta} f^-(x,t_{m-1})\phi_J^m dx+(1-\sigma)\int_{\zeta}^{x_{J+1}} f^+(x,t_{m-1})\phi_J^m dx & i=J\\ (1-\sigma)\int_{x_J}^{\zeta} f^-(x,t_{m-1})\phi_{J+1}^m dx+(1-\sigma)\int_{\zeta}^{x_{J+2}} f^+(x,t_{m-1})\phi_{J+1}^m dx & i=J+1\end{cases}.$$

The matrices A^3 and B^4 correspond to the corrections, results of the δ - Dirac function.

NUMERICAL EXPERIMENTS

As a test problem we consider

$$u_t(x,t)-(\beta u_x(x,t))_x=-\delta(x-\zeta)Ku(x,t)\ \ (x,t)\in(0,1)\times(0,1],$$

with initial and boundary conditions

$$u(x,0)=u_0(x)=\begin{cases}\dfrac{\cos(x/\sqrt{\beta^-})}{\cos(\zeta/\sqrt{\beta^-})},\ 0\le x\le\zeta\\ \dfrac{\sin(x/\sqrt{\beta^+})}{\sin(\zeta/\sqrt{\beta^+})},\ \zeta<x\le 1\end{cases}$$

$$u(0,t)=u_L(t)=\frac{\exp(-t)}{\cos(\zeta/\sqrt{\beta^-})},\ u(1,t)=u_R(t)=\frac{\sin(1/\sqrt{\beta^+})\exp(-t)}{\sin(\zeta/\sqrt{\beta^+})}.$$

The function $K(t)$ for this problem is a constant $K=\sqrt{\beta^+}ctg(\frac{\zeta}{\sqrt{\beta^+}})+\sqrt{\beta^-}tg(\frac{\zeta}{\sqrt{\beta^-}})$.

The jump conditions are: $$[u(x,t)]_{x=\zeta}=u(\zeta^+,t)-u(\zeta^-,t)=0,$$ $$[\beta u_x(x,t)]_{x=\zeta}=\beta^+u_x(\zeta^+,t)-\beta^-u_x(\zeta^-,t)=K(t)u(\zeta,t)=Ku(\zeta,t).$$

The exact solution is

$$u(x,t)=\begin{cases}\dfrac{\cos(x/\sqrt{\beta^-})}{\cos(\zeta/\sqrt{\beta^-})}\exp(-t), & 0\le x\le\zeta\\ \dfrac{\sin(x/\sqrt{\beta^+})}{\sin(\zeta/\sqrt{\beta^+})}\exp(-t), & \zeta<x\le 1\end{cases}.$$

In Table 1 the results for the parameters $\sigma=1/2, \zeta=\pi/6$, $K=\sqrt{\beta^+}ctg(\frac{\zeta}{\sqrt{\beta^+}})+\sqrt{\beta^-}tg(\frac{\zeta}{\sqrt{\beta^-}})$ are presented. We choose two different cases for the discontinuous coefficients $\beta^-=10, \beta^+=1$ and $\beta^-=1, \beta^+=10$. The error of the numerical solution in maximum norm for different N and M is denoted by $\| er_N^M \|_\infty=\max_{i,m}(|u(x_i,t_m)-z_m^h(x_i)|)$, and the rate of convergence - by $rate=\log_2(\frac{\| er_N^M \|_\infty}{\| er_{2N}^{2M} \|_\infty})$. The results confirm that the method is of second order on space and time for the case of $\sigma=1/2$, when the method of weights is known as Crank-Nicolson method.

In Figure 1 the exact solution and the error of the numerical solution for $N=M=40$ and
$\beta^-=1, \beta^+=10$, $\sigma=1/2, \zeta=\pi/6$ are presented.

Table 1. The numerical results for $\sigma=1/2, \zeta=\pi/6, K=\sqrt{\beta^+}ctg(\frac{\zeta}{\sqrt{\beta^+}})+\sqrt{\beta^-}tg(\frac{\zeta}{\sqrt{\beta^-}})$

N	M	$\beta^-=10, \beta^+=1$		$\beta^-=1, \beta^+=10$	
		$\| er_N^M \|_\infty$	$rate$	$\| er_N^M \|_\infty$	$rate$
5	5	0.000367	-	0.000436	-
10	10	8.3325e-5	2.1418	9.2664e-5	2.2364
20	20	1.6931e-5	2.2990	1.8992e-5	2.2865
40	40	4.0763e-6	2.0543	4.5944e-6	2.0474
80	80	1.0141e-6	2.0070	1.1397e-6	2.0112
160	160	2.5325e-7	2.0015	2.8474e-7	2.0009
320	320	6.3306e-8	2.0001	7.1157e-8	2.0005

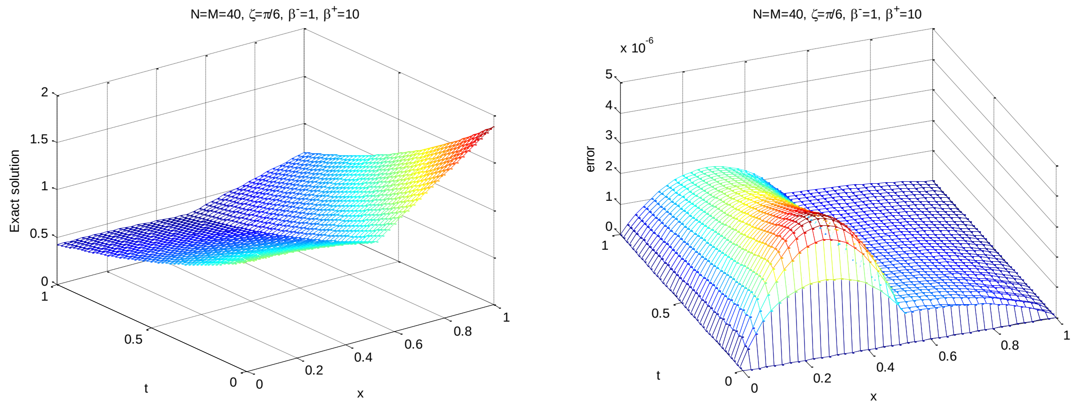

Fig.1. The exact solution and the error of the numerical solution for N=M=40 and $\beta^- = 1, \beta^+ = 10$.

Many other numerical experiments with different values of the coefficients have been done. All of them confirm the following proposition:

Proposition: *If the assumptions for the coefficients and functions are fulfilled, then the proposed IIFEM in the case* $\sigma = 1/2$ *is of second order both in space and time, i.e. for the error of the numerical solution of (9), (10) u (11)* z_m^h *the estimate.*

$$\| z_m^h - u \|_\infty \le C(h^2 + \tau^2)$$

holds, where the constant C *does not depend on* h *and* τ.

CONCLUSION

In this work we investigate the application of the IIFEM for a parabolic problem with local own sources on some interface, embedding in the domain. For the numerical solution we use method of weights and FEM with special basic functions satisfying the jump conditions on the interface. Second order of convergence in the case $\sigma = 1/2$ is numerically proved. The theoretical proof of the proposition and application of ODE solvers of Matlab are object of our forthcoming work.

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CONTACT ADDRESSES

Assist. Ivan Georgiev
Department of Applied Mathematics and Statistics
Faculty of Public Health and Health Care
Angel Kanchev University of Ruse
8 Studentska Str.,
7017 Ruse, Bulgaria
Phone: (++359 82) 888 424
E-mail: irgeorgiev@uni-ruse.bg

Assoc. Prof. Juri Kandilarov, PhD
Department of Mathematics
Faculty of Natural Sciences and Education
Angel Kanchev University of Ruse
8 Studentska Str.,
7017 Ruse, Bulgaria
Phone: (++359 82) 888 634
E-mail: ukandilarov@uni-ruse.bg

ВЛОЖЕН ИНТЕРФЕЙСЕН МЕТОД НА КРАЙНИТЕ ЕЛЕМЕНТИ ЗА УРАВНЕНИЕ НА ДИФУЗИЯТА С ЛОКАЛЕН ИЗТОЧНИК

Иван Георгиев, Юрий Кандиларов

Русенски университет "Ангел Кънчев"

***Резюме**: В статията се разглежда уравнение на дифузията с локален собствен източник. Дефинирана е слаба формулировка на задачата и е приложен вложен интерфейсен метод на крайните елементи за численото решаване на проблема. При дискретизацията по времето е използван метод на Роте с тегла. След това са въведени специални базисни функции, които удовлетворяват условията на скока на решението и потока върху интерфейса. Представени са числени експерименти, които потвърждават втори ред на точност в максимална норма.*

***Ключови думи:** вложен интерфейс, метод на крайните елементи, уравнение на дифузията, локални източници, метод на Роте*

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