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Union of Scientists - Ruse
16, Konstantin Irechek Street
7000 Ruse
BULGARIA
Phone: (++359 82) 828 135,
(++359 82) 841 634
E-mail: suruse@uni-ruse.bg
web: suruse.uni-ruse.bg

Contacts with Editor
Phone: (++359 82) 888 738
E-mail: zzdravkova@uni-ruse.bg

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MARGINAL DENSITIES OF THE WISHART DISTRIBUTION CORRESPONDING TO CYCLE GRAPHS[1]

Evelina Veleva

Angel Kanchev University of Ruse

Abstract: *The aim of the paper is to suggest a technique for the calculation of marginal densities of the Wishart distribution, corresponding to cycle graphs. Each non-decomposable graph consists of at least one cycle with 4 or more edges. Using the results for decomposable graphs, it is shown how by Monte Carlo method for numerical integration a marginal density, corresponding to any cyclic graph can be computed at a given point.*

Keywords: *Wishart distribution, non-decomposable graph, marginal density, covariance matrix, graphical Gaussian models*

INTRODUCTION

The distribution of the sample covariance matrix for a sample from a multivariate normal distribution is derived in 1928 by Wishart ([11]) and is known as the Wishart distribution. It is a multivariate generalization of the χ^2 (chi-square) distribution and is present in almost all textbooks on multivariate statistical analysis. Wishart distribution is subject to research and generalization since its introduction in 1928 until now. The study of this distribution is relevant and up to date as evidenced by the publication of more new materials on the subject (see e.g. [1], [6], [7]). A $p \times p$ random matrix with Wishart distribution $\mathrm{W}_p(n,\Sigma)$, where $p < n+1$ and Σ is a positive definite $p \times p$ matrix, has probability density of the form

$$f_{p,n,\Sigma}(\mathrm{W}) = \frac{1}{2^{np/2}\Gamma_p\left(n/2\right)(\det\Sigma)^{n/2}}(\det\mathrm{W})^{(n-p-1)/2}e^{-tr(\mathrm{W}\Sigma^{-1})/2} \qquad (1)$$

for every real $p \times p$ positive definite matrix W, where $\Gamma_p(\cdot)$ is the multivariate Gamma function defined as $\Gamma_p\left(\gamma\right) = \pi^{p(p-1)/4}\prod_{j=1}^{p}\Gamma[\gamma+(1-j)/2]$ and $\det(\cdot)$, $tr(\cdot)$ denote the determinant and the trace of a matrix. When the observed multivariate normal distributed random variables are mutually independent, Σ as well as its inverse Σ^{-1} are diagonal matrices. We shall denote the elements of the matrix Σ^{-1} by $\sigma^{i,j}$. The equality $\sigma^{i,j} = 0$ means that the corresponding pair of variables X_i and X_j are conditionally independent given all other $p-2$ variables. The presence of conditional independence between pairs of factors given the others underlies graphical Gaussian models ([5]). In some scientific fields, such as genetics, majority of pairs of factors are conditionally independent given the rest. The elements of the sample covariance matrix for which the elements of the matrix Σ^{-1} are non-zero form a set of sufficient statistics for the estimation of the covariance matrix Σ (see [2]). When modelling covariance matrices it is sufficient to include in the model only those covariance coefficients for which $\sigma^{i,j} \neq 0$. The respective marginal density of the

[1]This paper contains results of the work on project No 2015 - FPHHC – 03, financed by „Scientific Research" Fund of Ruse University.

Wishart distribution which will be used for the estimation of the parameters of the model can be obtained by integrating the density (1) with respect to those $w_{i,j}$ for which $\sigma^{i,j}=0$.

By integrating the density (1), some marginal densities of the Wishart distribution are derived in [8] in explicit form. The bounds of the integration however, cannot always be exactly obtained. Let $\mathbf{W}=(W_{i,j})$ be a $p\times p$ positive definite random matrix with an arbitrary distribution. Let M be a subset of the set $\{W_{i,j}, 1\le i<j\le p\}$ of non-diagonal elements of $\mathbf{W}$. The marginal density f_{M^C}, obtained after integration of the density function of $\mathbf{W}$ with respect to the variables from M, can be represented by a graph G_{-M} with the set of vertices $V=\{1,\dots,p\}$. For every element $W_{i,j}$ of the set $\{W_{i,j}, 1\le i<j\le p\}\setminus M$ we draw in the graph an undirected edge between the vertices „i" and „j". For instance, the graph on Figure 1 presents the joint (marginal) density of the random variables from the set $\{W_{1,2}, W_{2,3}, W_{3,4}, W_{4,5}, W_{5,6}, W_{6,7}, W_{7,8}, W_{1,8}\}$. For a graph with $p=8$ vertices, the number of all possible edges is $C_8^2=28$. In this case the set M consists of the rest 20 variables with respect to which the density function of $\mathbf{W}$ is integrated.

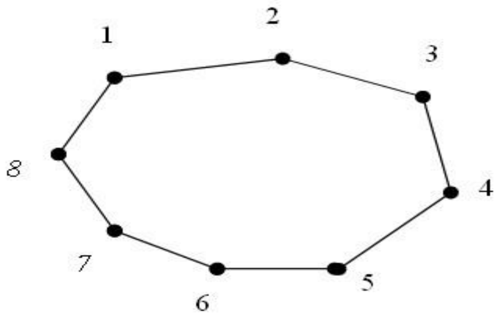

Fig.1. Cycle graph

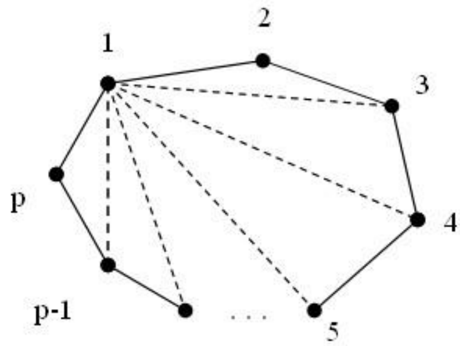

Fig.2. Decomposable graph

The next Proposition is proved in [9].

Proposition 1. *The bounds of the integration of the density function of* $\mathbf{W}$ *with respect to all the elements of* M *can be exactly obtained if and only if the corresponding graph* G_{-M} *is decomposable.*

We shall recall some basic concepts from graph theory (see [5]). Let $G=(V,E)$ be a graph with a finite set of vertices V and a set of undirected edges E. G is called a complete graph if every pair of distinct vertices is connected by an edge. A subset of vertices $U\subset V$ defines an induced subgraph G_U of G which contains all the vertices U and any edges in E that connect vertices in U. A clique is a complete subgraph that is maximal, that is, it is not a subgraph of any other complete subgraph.

Definition 1. *A graph* G *is decomposable if and only if the set of cliques of* G *can be ordered as* $(C_1,\ C_2,\dots,\ C_k)$ *so that for each* $i=2,\dots,k$ *if* $S_i=C_i\bigcap\left(\bigcup_{j=1}^{i-1}C_j\right)$ *then* $S_i\subset C_l$ *for some* $l<i$.

For a decomposable graph G_{-M}, if $\sigma^{i,j}=0$ for every element $W_{i,j}$ of M, the marginal density f_{M^C} of the Wishart distribution has a compact form, derived in [9] and given below by Proposition 2. For an arbitrary $p\times p$ matrix A and a subset $\alpha\subset\{1,2,\dots,p\}$ we shall denote by $\mathrm{A}[\alpha]$ the submatrix of A, composed of the rows and columns with numbers from α. By $|\alpha|$ we denote the number of elements of a set α.

Proposition 2. *Let* $\mathbf{W}=(W_{i,j})$ *has Wishart distribution* $\mathrm{W}_p(n,\Sigma)$ *and* M *be a subset of the set* $\{W_{i,j}, 1\le i<j\le p\}$ *of non-diagonal elements of* $\mathbf{W}$. *Let* $\sigma^{i,j}=0$ *for every element* $W_{i,j}$ *of* M. *Let the graph* G_{-M} be decomposable *with* k *cliques* C_1, C_2,..., C_k, *ordered according to Definition 1 and* V_i *be the set of vertices of* C_i, $i=1,\dots,k$. *Then the joint density* f_{M^C} *of the elements of the set* $M^C=\{W_{i,j}, 1\le i\le j\le p\}\setminus M$ *can be written in the form*

$$f_{M^C}(\mathrm{W}_0)=\frac{\prod_{i=1}^{k} f_{|V_i|,n,\Sigma[V_i]}(\mathrm{W}_0[V_i])}{\prod_{i=2}^{k} f_{|U_i|,n,\Sigma[U_i]}(\mathrm{W}_0[U_i])}, \tag{2}$$

where $\mathrm{W}_0=(w_{i,j})$ *is a* $p\times p$ *symmetric matrix such that* $w_{i,j}=0$ *for* $W_{i,j}\in M$; $f_{p,n,\Sigma}(\mathrm{W})$ *is the Wishart density function given by (1) and* $U_i=(V_1\cup\ldots\cup V_{i-1})\cap V_i$, $i=2,\dots,k$.

Obtaining of marginal densities corresponding to non-decomposable graphs is discussed in [8]. According to Proposition 1, they cannot be expressed in explicit form. For a non-decomposable graph G_{-M} the marginal density f_{M^C} can be calculated numerically for every set of fixed values for the random variables from the set $M^C=\{W_{i,j}, 1\le i\le j\le p\}\setminus M$. These values can be written in the form of a matrix W_0, where the elements corresponding to the elements of M are zeros. For the calculation of $f_{M^C}(\mathrm{W}_0)$ in [10], the using of Monte Carlo method for numerical integration is suggested. Let M_1 and M_2 be sets such that

1) $M_1\subset M\subset M_2\subset\{W_{i,j}, 1\le i<j\le p\}$ and the densities $f_{M_1^C}$ and $f_{M_2^C}$ correspond to decomposable graphs;
2) the density $f_{M_2^C\cup L}$, where $L=M\setminus M_1$, corresponds to a decomposable graph.

The choice of M_1 and M_2 is not unique and is always possible. Then $f_{M^C}(\mathrm{W}_0)$ can be written as

$$f_{M^C}(\mathrm{W}_0)=\int_D\left(\frac{f_{M_1^C}(\mathrm{W}_0,\mathrm{x})}{g(\mathrm{W}_0,\mathrm{x})}\right)g(\mathrm{W}_0,\mathrm{x})\,d\mathrm{x}, \tag{3}$$

where D is the domain of definition of $f_{M_1^C}$, the integration is with respect to all variables from the set L and $g(\mathrm{W}_0,\mathrm{x})$ is a probability density function on D. In [10], $g(\mathrm{W}_0,\mathrm{x})$ is proposed to be the conditional probability density function

$$g(\mathrm{W}_0,\mathrm{x})=f_{L/M_2^C}(\mathrm{W}_0,\mathrm{x})=\frac{f_{M_2^C\cup L}(\mathrm{W}_0,\mathrm{x})}{f_{M_2^C}(\mathrm{W}_0)}, \tag{4}$$

derived by Proposition 2 under the condition that $\sigma^{i,j}=0$ for every element $W_{i,j}$ of M_2. By (3), $f_{M^C}(\mathrm{W}_0)$ is presented as the expectation of the random variable

$$\frac{f_{M_1^C}(\mathrm{W}_0,\xi)}{g(\mathrm{W}_0,\xi)}, \tag{5}$$

where ξ is a random vector with density function $g(\mathrm{W}_0,\mathrm{x})$. Hence $f_{M^C}(\mathrm{W}_0)$ can be estimated by the mean value of a large number of realizations of (5). The main difficulty therefore is the generation of realizations of a random vector ξ with a given probability density function $g(\mathrm{W}_0,\mathrm{x})$. Each non-decomposable graph consists of at least one cycle with 4 or more edges. The aim of this paper is to be applied the proposed approach for the calculation of marginal densities of the Wishart distribution, corresponding to cycle graphs with any number of edges, greater than 3.

MARGINAL DENSITIES CORRESPONDING TO CYCLE GRAPHS

Let $\mathbf{W}$ be a random matrix with Wishart distribution $\mathrm{W}_p(n,\Sigma)$. Let us consider the joint density of the variables from the set $M^C=\{W_{1,2},W_{2,3},W_{3,4},\ldots,W_{p-1,p},W_{1,p},W_{1,1},\ldots,W_{p,p}\}$, presented by the continuous edges on the graph of Figure 2. We have to choose the sets M_1 and M_2, $M_1\subset M\subset M_2\subset\{W_{i,j},1\le i<j\le p\}$, such that the densities $f_{M_1^C}$ and $f_{M_2^C}$ correspond to decomposable graphs. The density $f_{M_2^C\cup L}$, where $L=M\setminus M_1$, also have to correspond to a decomposable graph. This conditions will be satisfied if we choose $L=\{W_{1,3},W_{1,4},W_{1,5},\ldots,W_{1,p-1}\}$ so that M_1^C is presented by all edges on the graph of Figure 2 and $M_2^C=M^C\setminus\{W_{1,p}\}$. Let $y_{1,2},y_{2,3},\ldots,y_{p-1,p},y_{1,p},y_{1,1},\ldots y_{p,p}$ be fixed values for the random variables from the set M^C. We suppose that $y_{i,j}>0$ for $i=j$ and $y_{i,i}y_{j,j}>y_{i,j}^2$ for $i\ne j$, otherwise these values cannot be elements of a positive definite matrix. Let us denote by W_0 the matrix $\mathrm{W}_0=(w_{i,j}^0)$ such that

$$w_{i,j}^0=\begin{cases} y_{i,j} & if\ W_{i,j}\in M^C\ or\ W_{j,i}\in M^C \\ 0 & otherwise\end{cases}.$$

The variables $x_{1,3},x_{1,4},\ldots,x_{1,p-1}$ correspond to the elements of the set L with respect to which we have to integrate the density $f_{M_1^C}$. Let W_x be the matrix $\mathrm{W}_x=(w_{i,j}^x)$ such that

$$w_{i,j}^x=\begin{cases} x_{i,j} & if\ W_{i,j}\in L\ or\ W_{j,i}\in L \\ w_{i,j}^0 & otherwise\end{cases}.$$

The graph G_{-M_1}, shown on Figure 2, is decomposable with $p-2$ cliques C_i, $i=1,\ldots,p-2$ with sets of vertices $V_i=\{1,i+1,i+2\}$, $i=1,\ldots,p-2$. Assuming $\sigma^{i,j}=0$ for $W_{i,j}\in M_1$, according to Proposition 2, the density $f_{M_1^C}$ has the form

$$f_{M_1^C}(\mathrm{W}_x)=\frac{\prod_{i=1}^{p-2} f_{3,n,\Sigma[V_i]}(\mathrm{W}_x[V_i])}{\prod_{i=2}^{p-2} f_{2,n,\Sigma[U_i]}(\mathrm{W}_x[U_i])},$$

where $U_i=\{1,i+1\}$, $i=2,\ldots,p-2$.

The probability density function $g(\mathrm{W}_x)=g(\mathrm{W}_0,\mathrm{x})$ will be computed under the condition that $\sigma^{i,j}=0$ for every element $W_{i,j}$ of M_2. The graph G_{-M_2} is decomposable with $p-1$ cliques with sets of vertices $T_i=\{i,i+1\}$, $i=1,\ldots,p-1$ respectively. Therefore, according to Proposition 2

$$f_{M_2^C}(\mathrm{W}_0)=\frac{\prod_{i=1}^{p-1} f_{2,n,\Sigma[T_i]}(\mathrm{W}_0[T_i])}{\prod_{i=2}^{p-1} f_{1,n,\Sigma[\{i\}]}(\mathrm{W}_0[\{i\}])}.$$

The graph, corresponding to the marginal density $f_{M_2{}^C\cup L}$ consists of all edges on Figure 2 except the edge connecting the vertices 1 and p. This graph is decomposable with $p-2$ cliques with sets of vertices V_i, $i=1,\dots,p-3$ and T_{p-1} respectively. Consequently

$$f_{M_2^C\cup L}(\mathrm{W}_x)=\frac{f_{2,n,\Sigma[T_{p-1}]}(\mathrm{W}_x[T_{p-1}])\prod_{i=1}^{p-3} f_{3,n,\Sigma[V_i]}(\mathrm{W}_x[V_i])}{f_{1,n,\Sigma[\{p-1\}]}(\mathrm{W}_x[\{p-1\}])\prod_{i=2}^{p-3} f_{2,n,\Sigma[U_i]}(\mathrm{W}_x[U_i])},$$

hence

$$g(\mathrm{W}_x)=f_{L/M_2^C}(\mathrm{W}_x)=\frac{f_{M_2^C\cup L}(\mathrm{W}_x)}{f_{M_2^C}(\mathrm{W}_0)}=\frac{\left(\prod_{i=1}^{p-3} f_{3,n,\Sigma[V_i]}(\mathrm{W}_x[V_i])\right)\left(\prod_{i=2}^{p-2} f_{1,n,\Sigma[\{i\}]}(\mathrm{W}_x[\{i\}])\right)}{\left(\prod_{i=2}^{p-3} f_{2,n,\Sigma[U_i]}(\mathrm{W}_x[U_i])\right)\left(\prod_{i=1}^{p-2} f_{2,n,\Sigma[T_i]}(\mathrm{W}_x[T_i])\right)}. \tag{6}$$

Each realization of the random variable (5) will have the form

$$\frac{f_{M_1^C}(\mathrm{W}_x)}{g(\mathrm{W}_x)}=\left(\frac{\prod_{i=1}^{p-2} f_{2,n,\Sigma[T_i]}(\mathrm{W}_0[T_i])}{\prod_{i=2}^{p-2} f_{1,n,\Sigma[\{i\}]}(\mathrm{W}_0[\{i\}])}\right)\frac{f_{3,n,\Sigma[V_{p-2}]}(\mathrm{W}_x[V_{p-2}])}{f_{2,n,\Sigma[U_{p-2}]}(\mathrm{W}_x[U_{p-2}])}. \tag{7}$$

For the calculation of (7) we must have a realization $\boldsymbol{x}=\{x_{1,3},x_{1,4},\dots,x_{1,p-1}\}$ of a random vector $\boldsymbol{\xi}=\{\xi_{1,3},\xi_{1,4},\dots,\xi_{1,p-1}\}$ with density function (6). Let us consider the transformation

$$x_{1,3}=\frac{1}{y_{2,2}}\left[y_{1,2}y_{2,3}+z_{1,3}\sqrt{(y_{1,1}y_{2,2}-y_{1,2}^2)(y_{2,2}y_{3,3}-y_{2,3}^2)}\right], \tag{8}$$

$$x_{1,4}=\frac{1}{y_{3,3}}\left[x_{1,3}y_{3,4}+z_{1,4}\sqrt{(y_{1,1}y_{3,3}-x_{1,3}^2)(y_{3,3}y_{4,4}-y_{3,4}^2)}\right], \tag{9}$$

$$x_{1,p-1}=\frac{1}{y_{p-2,p-2}}\left[x_{1,p-2}y_{p-2,p-1}+z_{1,p-1}\sqrt{(y_{1,1}y_{p-2,p-2}-x_{1,p-2}^2)(y_{p-2,p-2}y_{p-1,p-1}-y_{p-2,p-1}^2)}\right]. \tag{10}$$

The inverse transformation is

$$z_{1,3}=\frac{x_{1,3}y_{2,2}-y_{1,2}y_{2,3}}{\sqrt{(y_{1,1}y_{2,2}-y_{1,2}^2)(y_{2,2}y_{3,3}-y_{2,3}^2)}},$$

$$z_{1,4}=\frac{x_{1,4}y_{3,3}-x_{1,3}y_{3,4}}{\sqrt{(y_{1,1}y_{3,3}-x_{1,3}^2)(y_{3,3}y_{4,4}-y_{3,4}^2)}},$$

$$z_{1,p-1}=\frac{x_{1,p-1}y_{p-2,p-2}-x_{1,p-2}y_{p-2,p-1}}{\sqrt{(y_{1,1}y_{p-2,p-2}-x_{1,p-2}^2)(y_{p-2,p-2}y_{p-1,p-1}-y_{p-2,p-1}^2)}}.$$

It is easy to see that all elements above the main diagonal in the Jacobian $\mathbf{J}$, $\mathbf{J}=\dfrac{\partial\left(z_{1,3},z_{1,4},\dots,z_{1,p-1}\right)}{\partial\left(x_{1,3},x_{1,4},\dots,x_{1,p-1}\right)}$ are equal to zero. Consequently

$$\det \mathbf{J} = \frac{\partial z_{1,3}}{\partial x_{1,3}} \frac{\partial z_{1,4}}{\partial x_{1,4}} \cdots \frac{\partial z_{1,p-1}}{\partial x_{1,p-1}}$$

$$= \frac{y_{2,2}}{\sqrt{(y_{1,1}y_{2,2} - y_{1,2}^2)(y_{2,2}y_{3,3} - y_{2,3}^2)}} \frac{y_{3,3}}{\sqrt{(y_{1,1}y_{3,3} - x_{1,3}^2)(y_{3,3}y_{4,4} - y_{3,4}^2)}} \times \cdots$$

$$\times \frac{y_{p-2,p-2}}{\sqrt{(y_{1,1}y_{p-2,p-2} - x_{1,p-2}^2)(y_{p-2,p-2}y_{p-1,p-1} - y_{p-2,p-1}^2)}} = \frac{\prod_{i=2}^{p-2} \det(\mathrm{W}_x[\{i\}])}{\left(\prod_{i=1}^{p-2} \det(\mathrm{W}_x[T_i])\right)^{1/2} \left(\prod_{i=2}^{p-3} \det(\mathrm{W}_x[U_i])\right)^{1/2}}.$$

Theorem 1. Let $z_{1,3}, z_{1,4}, \ldots, z_{1,p-1}$ be realizations of independent and identically distributed random variables $\zeta_{1,3}, \zeta_{1,4}, \ldots, \zeta_{1,p-1}$ with density function

$$\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n-2}{2}\right)\Gamma\left(\frac{1}{2}\right)} (1-z^2)^{\frac{n-4}{2}}, \qquad z \in (-1,1). \tag{11}$$

Then the variables $x_{1,3}, x_{1,4}, \ldots, x_{1,p-1}$ defined by equalities (8) – (10) are realizations of random variables $\xi_{1,3}, \xi_{1,4}, \ldots, \xi_{1,p-1}$ with joint density function (6).

Proof. The joint density function of the random variables $\zeta_{1,3}, \zeta_{1,4}, \ldots, \zeta_{1,p-1}$ is

$$f_\zeta(z_{1,3}, z_{1,4}, \ldots, z_{1,p-1}) = \left[\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n-2}{2}\right)\Gamma\left(\frac{1}{2}\right)}\right]^{p-3} (1-z_{1,3}^2)^{\frac{n-4}{2}} (1-z_{1,4}^2)^{\frac{n-4}{2}} \ldots (1-z_{1,p-1}^2)^{\frac{n-4}{2}},$$

for $z_{1,i} \in (-1,1)$, $i = 3, \ldots, p-1$. Using (8), it is easy to check that

$$\det\begin{pmatrix} y_{1,1} & y_{1,2} & x_{1,3} \\ y_{1,2} & y_{2,2} & y_{2,3} \\ x_{1,3} & y_{2,3} & y_{3,3} \end{pmatrix} = \frac{1}{y_{2,2}} (y_{1,1}y_{2,2} - y_{1,2}^2)(y_{2,2}y_{3,3} - y_{2,3}^2)(1 - z_{1,3}^2).$$

Consequently $1 - z_{1,3}^2 = \dfrac{y_{2,2} \det(\mathrm{W}_x[V_1])}{\det(\mathrm{W}_x[T_1]) \det(\mathrm{W}_x[T_2])}$.

Analogously, for $i = 2, \ldots, p-3$

$$\det\begin{pmatrix} y_{1,1} & x_{1,i+1} & x_{1,i+2} \\ x_{1,i+1} & y_{i+1,i+1} & y_{i+1,i+2} \\ x_{1,i+2} & y_{i+1,i+2} & y_{i+2,i+2} \end{pmatrix} = \frac{1}{y_{i+1,i+1}} (y_{1,1}y_{i+1,i+1} - x_{1,i+1}^2)(y_{i+1,i+1}y_{i+2,i+2} - y_{i+1,i+2}^2)(1 - z_{1,i+2}^2)$$

and hence for $i = 2, \ldots, p-3$

$$1 - z_{1,i+2}^2 = \frac{y_{i+1,i+1} \det(\mathrm{W}_x[V_i])}{\det(\mathrm{W}_x[U_i]) \det(\mathrm{W}_x[T_{i+1}])}.$$

The joint density of the random variables $\xi_{1,3}, \xi_{1,4}, \ldots, \xi_{1,p-1}$ gets the form

$$f_\xi(x_{1,3}, x_{1,4}, \ldots, x_{1,p-1}) = \left[\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n-2}{2}\right)\Gamma\left(\frac{1}{2}\right)}\right]^{p-3} \det \mathbf{J} \times$$

$$\left[\frac{y_{2,2}\det(W_x[V_1])}{\det(W_x[T_1])\det(W_x[T_2])}\cdot\frac{y_{3,3}\det(W_x[V_2])}{\det(W_x[U_2])\det(W_x[T_3])}\cdots\frac{y_{p-2,p-2}\det(W_x[V_{p-3}])}{\det(W_x[U_{p-3}])\det(W_x[T_{p-2}])}\right]^{\frac{n-4}{2}}=$$

$$=\left[\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n-2}{2}\right)\Gamma\left(\frac{1}{2}\right)}\right]^{p-3}\frac{\left(\prod_{i=2}^{p-2}\det(W_0[\{i\}])\right)^{\frac{n-2}{2}}}{\left(\prod_{i=1}^{p-2}\det(W_0[T_i])\right)^{\frac{n-3}{2}}}\cdot\frac{\left(\prod_{i=1}^{p-3}\det(W_x[V_i])\right)^{\frac{n-4}{2}}}{\left(\prod_{i=2}^{p-3}\det(W_x[U_i])\right)^{\frac{n-3}{2}}}.$$

With respect to the variables $x_{1,3},x_{1,4},\ldots,x_{1,p-1}$ the density function $g(W_x)$, given by (6), can be written in the form

$$g(W_x)=C\frac{\left(\prod_{i=1}^{p-3}\det(W_x[V_i])\right)^{\frac{n-4}{2}}}{\left(\prod_{i=2}^{p-3}\det(W_x[U_i])\right)^{\frac{n-3}{2}}},$$

if the matrices $W_x[V_i]$ are positive definite. Here C is a function of the values $y_{1,2},y_{2,3},\ldots,y_{p-1,p},y_{1,p},y_{1,1},\ldots y_{p,p}$ but does not depend on $x_{1,3},x_{1,4},\ldots,x_{1,p-1}$. This completes the proof.

The distribution with density function (11) is known as Pearson distribution of the second type (see [4]) and also as power semicircle distribution (see [3]). Random variables with this distribution can be easily generated (see [4], p. 481) using the quotient of the difference and the sum of two Gamma distributed random variables.

For every $p-3$ realizations $z_{1,3},z_{1,4},\ldots,z_{1,p-1}$ of a random variable with density function (11) by formulas (8) – (10) we get a realization $\boldsymbol{x}=\{x_{1,3},x_{1,4},\ldots,x_{1,p-1}\}$ of a random vector $\xi=\{\xi_{1,3},\xi_{1,4},\ldots,\xi_{1,p-1}\}$ with density function (6).

If $\boldsymbol{x}_1$, …, $\boldsymbol{x}_N$ are N realizations of ξ then according to (7)

$$f_{M^C}(W_0)\approx\left(\frac{\prod_{i=1}^{p-2}f_{2,n,\Sigma[T_i]}(W_0[T_i])}{\prod_{i=2}^{p-2}f_{1,n,\Sigma[\{i\}]}(W_0[\{i\}])}\right)\frac{1}{N}\sum_{i=1}^{N}\frac{f_{3,n,\Sigma[V_{p-2}]}(W_{\boldsymbol{x}_i}[V_{p-2}])}{f_{2,n,\Sigma[U_{p-2}]}(W_{\boldsymbol{x}_i}[U_{p-2}])}.$$

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CONTACT ADDRESS
Assoc. Prof. Evelina Veleva, PhD
Department of Applied Mathematics and Statistics
Angel Kanchev University of Ruse
8, Studentska Str., 7017 Ruse, Bulgaria
Phone: (++359 82) 888 606
E-mail: eveleva@uni-ruse.bg

МАРГИНАЛНИ ПЛЪТНОСТИ НА РАЗПРЕДЕЛЕНИЕТО НА УИШАРТ, СЪОТВЕТСТВАЩИ НА ЦИКЛИЧНИ ГРАФИ

Евелина Велева

Русенски университет “Ангел Кънчев”

***Резюме**: Целта на статията е да предложи техника за изчисляване на маргиналните плътности на разпределението на Уишарт, съответстващи на циклични графи. Всеки неразложим граф се състои от поне един цикъл с 4 или повече дъги. Използвайки резултатите за разложими графи е показано как с помощта на метода Монте Карло за числено интегриране, маргинална плътност, съответстваща на произволен цикличен граф, може да бъде пресметната в зададена точка.*

***Ключови думи:** разпределение на Уишарт, неразложим граф, маргинална плътност, ковариационна матрица, графични Гаусови модели.*

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